Flag vectors of Eulerian partially ordered sets

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Abstract

The closed cone of flag vectors of Eulerian partially ordered sets is studied. A new family of linear inequalities valid for Eulerian flag vectors is given. Half-Eulerian posets are defined. Certain limit posets of Billera and Hetyei are half-Eulerian; they give rise to extreme rays of the cone for Eulerian posets. Other extreme posets are formed from consideration of the *cd*-index. The cone of Eulerian flag vectors is completely determined up through rank seven.

1 Introduction

The study of Eulerian partially ordered sets (posets) originated with Stanley ([18]). Examples of Eulerian posets are the posets of faces of regular CW spheres. These include face lattices of convex polytopes, the Bruhat order on finite Coxeter groups, and the lattices of regions of oriented matroids. (See [11] and [12].)

The flag f-vector (or simply flag vector) of a poset is a standard parameter counting chains in the partially ordered set by ranks. In the last twenty years there has grown a body of work on numerical conditions on flag vectors of posets and complexes, especially those arising in geometric contexts. Early contributions are from Stanley on balanced Cohen-Macaulay complexes ([17]) and Bayer and Billera on the linear equations on flag vectors of Eulerian posets ([4]). For an extensive survey of inequalities on flag

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numbers of polytopes see [16]. A major recent contribution is the determination of the closed cone of flag vectors of all graded posets by Billera and Hetyei ([8]). Results on flag vectors and other invariants of Eulerian posets and special classes of them are surveyed in [21].

Our goal has been to describe the closed cone of flag f-vectors of Eulerian partially ordered sets. This problem was posed explicitly in [10]. There is no reason to expect that every positive integer vector in this cone is the flag vector of some Eulerian poset. Nonlinear inequalities may come into play, but their analysis is much more difficult. We focus here on linear inequalities valid for all Eulerian flag vectors and the Eulerian posets with extreme flag vectors. This approach has been used previously to study f-vectors and flag vectors of various classes of posets. See Bayer ([2]) on four-dimensional polytopes, Babson, Billera and Chan ([1]) on cubical polytopes, and Billera and Hetyei ([8, 9]) on graded posets and planar posets. In all cases where the cone is known, it turns out to be finitely generated; this is verified only by finding a complete, finite set of facets or extremes. We expect that the same holds for the cone of flag vectors of Eulerian posets.

Finding the facets of the cone means finding all crucial inequalities satisfied by Eulerian flag vectors. One set of inequalities (given in Proposition 3.1) follows easily from the definition of Eulerian. A second set proved here (Theorem 3.2) generalizes an inequality found by Billera and Liu ([10]). We do not know if these two classes of inequalities are enough to determine completely the cone.

Billera and Hetyei ([8]) developed a poset construction that yields all the extreme rays of the cone of flag vectors of graded posets. By introducing the concept of a "half-Eulerian" poset, we are able to use the Billera-Hetyei construction to find extreme rays of the Eulerian cone. For rank $n+1$ this gives $\binom{n}{\lfloor n/2 \rfloor}$ extreme rays. Not all the extreme rays come this way, however, and we are forced to use more complicated constructions of extreme Eulerian posets. These constructions are suggested by the ce-index (a variation of the cd-index).

The ability to explore the cone in low ranks using the computer package PORTA ([14]) was crucial to this project. The straightforward description of the cone in ranks at most 6 breaks down at rank 7. This leads us to new classes of inequalities and extremes.

The remainder of this section provides definitions and other background, and the definition of the flag L-vector, which simplifies the calculations. Section 2 describes the extreme rays of the general graded cone, defines half-Eulerian posets, identifies which limit posets are half-Eulerian, and computes the corresponding cd-indices. Section 3 gives two general classes of inequalities on Eulerian flag vectors. Section 4 shows that the half-Eulerian limit posets all give extremes of the Eulerian cone, identifies some inequalities in all ranks as facet-inducing, and describes completely the cone for rank at most 7.

1.1 Background

A *graded poset* P is a finite partially ordered set with a unique minimum element $\hat{0}$, a unique maximum element $\hat{1}$, and a *rank function* $\rho : P \longrightarrow \mathbf{N}$ satisfying $\rho(\hat{0}) = 0$, and $\rho(y) - \rho(x) = 1$ whenever $y \in P$ covers $x \in P$. The *rank* $\rho(P)$ *of a graded poset* P is the rank of its maximum element. Given a graded poset P of rank $n+1$ and a subset S of $\{1, 2, \ldots, n\}$ (which we abbreviate as $[1, n]$), define the *S–rank–selected subposet of* P to be the poset

$$P_S = \{x \in P \ : \ \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}.$$

Denote by $f_S(P)$ the number of maximal chains of P_S. Equivalently, $f_S(P)$ is the number of chains $x_1 < \cdots < x_{|S|}$ in P such that $\{\rho(x_1), \ldots, \rho(x_{|S|})\} = S$. The vector $(f_S(P) \ : \ S \subseteq [1, n])$ is called the *flag f-vector* of P. Whenever it does not cause confusion, we write $f_{s_1 \ldots s_k}$ rather than $f_{\{s_1, \ldots, s_k\}}$; in particular, $f_{\{m\}}$ is always denoted f_m.

Various properties of the flag f-vector are more easily seen in different bases. An often used equivalent encoding is the *flag h-vector* $(h_S(P) \ : \ S \subseteq [1, n])$ given by the formula

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P),$$

or, equivalently,

$$f_S(P) = \sum_{T \subseteq S} h_T(P).$$

The *ab-index* $\Psi_P(a, b)$ of P is a generating function for the flag h-vector. It is the following polynomial in the noncommuting variables a and b:

$$\Psi_P(a, b) = \sum_{S \subseteq [1, n]} h_S(P) u_S, \tag{1}$$

where u_S is the monomial $u_1 u_2 \cdots u_n$ with $u_i = a$ if $i \notin S$, and $u_i = b$ if $i \in S$.

The *Möbius function* of a graded poset P is defined recursively for any subinterval of P by the formula

$$\mu([x, y]) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu([x, z]) & \text{otherwise.} \end{cases}$$

Equivalently, by Philip Hall's theorem, the Möbius function of a graded poset P of rank $n+1$ is the *reduced Euler characteristic* of the order complex, i.e., it is given by the formula

$$\mu(P) = \sum_{S \subseteq [1,n]} (-1)^{|S|+1} f_S(P). \tag{2}$$

(See [19, Proposition 3.8.5].)

A graded poset P is *Eulerian* if the Möbius function of every interval $[x, y]$ is given by $\mu([x, y]) = (-1)^{\rho(x,y)}$. (Here $\rho(x, y) = \rho([x, y]) = \rho(y) - \rho(x)$.)

The first characterization of all linear equalities holding for the flag f-vectors of all Eulerian posets was given by Bayer and Billera in [4]. The equations of the theorem are called the generalized Dehn-Sommerville equations. Call the subspace of $\mathbf{R}^{2^n}$ they determine the *Eulerian subspace*; its dimension is the Fibonacci number e_n ($e_0 = e_1 = 1$, $e_n = e_{n-1} + e_{n-2}$).

Theorem 1.1 (Bayer and Billera) *Every linear equality holding for the flag f-vector of all Eulerian posets of rank $n+1$ is a consequence of the equalities*

$$\left((-1)^{i-1} + (-1)^{k+1}\right) f_S + \sum_{j=i}^{k} (-1)^j f_{S \cup \{j\}} = 0$$

for $S \subseteq [1, n]$ and $[i, k]$ a maximal interval of $[1, n] \setminus S$.

Fine discovered that the ab-index of a polytope can be written as a polynomial in the noncommuting variables $c = a + b$ and $d = ab + ba$. Bayer and Klapper ([6]) proved that for a graded poset P, the equations of Theorem 1.1 hold if and only if the ab-index is a polynomial with integer coefficients in c and d. This polynomial is called the *cd-index* of P. Stanley ([20]) gives an explicit recursion for the cd-index in terms of intervals of P for Eulerian posets. (He thus gives another proof of the existence of the cd-index for Eulerian posets.)

1.2 The flag ℓ-vector and the flag L-vector

The introduction of another vector equivalent to the flag f-vector simplifies calculations.

Definition 1 The *flag ℓ-vector* of a graded partially ordered set P of rank $n+1$ is the vector $(\ell_S(P) : S \subseteq [1, n])$, where

$$\ell_S(P) = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} (-1)^{|T|} f_T(P).$$

As a consequence,

$$f_S(P) = \sum_{T \subseteq [1,n] \setminus S} \ell_T(P). \tag{3}$$

The flag ℓ-vector was first considered by Billera and Hetyei ([8]) while describing all linear inequalities holding for the flag f-vectors of all graded partially ordered sets. It turned out to give a sparse representation of the cone of flag f-vectors described in that paper.

A variant significant for Eulerian posets is the flag L-vector.

Definition 2 The *flag L-vector* of a graded partially ordered set P of rank $n+1$ is the vector $(L_S(P) \,:\, S \subseteq [1,n])$, where

$$L_S(P) = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} \left(-\frac{1}{2}\right)^{|T|} f_T(P).$$

Inverting the relation of the definition gives

$$f_S(P) = 2^{|S|} \sum_{T \subseteq [1,n] \setminus S} L_T(P).$$

When the poset P is Eulerian, the parameters $L_S(P)$ are actually the coefficients of the *ce-index* of the poset P. The ce-index was introduced by Stanley ([20]) as an alternative way of viewing the cd-index. The letter c continues to stand for $a+b$; now let $e = a-b$. The ab-index of a poset can be written in terms of c and d if and only if it can be written in terms of c and ee. It is easy to verify that $L_S(P)$ is exactly the coefficient in the *ce-index* of P of the word $u_S = u_1 u_2 \cdots u_n$ where $u_i = c$ if $i \notin S$, and $u_i = e$ if $i \in S$. Since the existence of the cd-index is equivalent to the validity of the generalized Dehn-Sommerville equations, we get the following proposition. (It can be proved directly from the definition of the flag L-vector, yielding an alternative way to prove the existence of the cd-index for Eulerian posets.) A subset $S \subseteq [1,n]$ is *even* if all the maximal intervals contained in S are of even length.

Proposition 1.2 *The generalized Dehn-Sommerville relations hold for a poset P if and only if $L_S(P) = 0$ whenever S is not an even set.*

The generalized Dehn-Sommerville relations hold (by chance) for some nonEulerian posets. A poset is Eulerian, however, if these relations hold for all intervals of the poset.

Corollary 1.3 *A graded partially ordered set is Eulerian if and only if $L_S([x,y]) = 0$ for every interval $[x,y] \subseteq P$ and every subset S of $[1, \rho(x,y)-1]$ that is not an even set.*

2 Half-Eulerian posets

In this section we find special points in the closed cone of flag vectors of Eulerian posets. First consider the extremes of the closed cone of flag vectors of all graded posets, found by Billera and Hetyei ([8]).

Definition 3 Given a graded poset P of rank $n+1$, an interval $I \subseteq [1,n]$, and a positive integer k, $D_I^k(P)$ is the graded poset obtained from P by replacing every $x \in P$ with rank in I by k elements $x_1, \ldots, x_k$ and by imposing the following relations.

(*i*) If for $x, y \in P$, $\rho(x) \in I$ and $\rho(y) \notin I$, then $x_i < y$ in $D_I^k(P)$ if and only if $x < y$ in P, and $y < x_i$ in $D_I^k(P)$ if and only if $y < x$ in P.

(*ii*) If $\{\rho(x), \rho(y)\} \subseteq I$, then $x_i < y_j$ in $D_I^k(P)$ if and only if $i = j$ and $x < y$ in P.

Clearly $D_I^k P$ is a graded poset of the same rank as P. Its flag f-vector can be computed from that of P in a straightforward manner.

An *interval system on* $[1,n]$ is any set of subintervals of $[1,n]$ that form an antichain (that is, no interval is contained in another). (Much of what follows holds even if the intervals do not form an antichain, but the assumption simplifies the statements of some theorems.) For any interval system $\mathcal{I}$ on $[1,n]$, and any positive integer N, the poset $P(n,\mathcal{I},N)$ is defined to be the poset obtained from a chain of rank $n+1$ by applying D_I^N for all $I \in \mathcal{I}$. It does not matter in which order these operators are applied. (Different values of N can be used for each interval I, but we do not need that generality here.) Consider the sequence of posets for a fixed interval system $\mathcal{I}$ as N goes to infinity. Billera and Hetyei ([8]) showed that the normalized flag vectors of such a sequence converge to a vector on an extreme ray of the cone of flag vectors of all graded posets. More precisely,

Theorem 2.1 (Billera and Hetyei) *Suppose $\mathcal{I}$ is an interval system of k intervals on $[1,n]$. Then the vector*

$$\left(\lim_{N\to\infty} \frac{1}{N^k} f_S(P(n,\mathcal{I},N)) : S \subseteq [1,n]\right)$$

generates an extreme ray of the cone of flag vectors of all graded posets of rank $n+1$. Moreover, all extreme rays are generated in this way.

Unfortunately, none of the posets $P(n,\mathcal{I},N)$ are Eulerian, and none of these extreme rays are contained in the closed cone of flag vectors of

Eulerian posets. However some of the posets are "half-Eulerian", and lead us to extreme rays of the Eulerian cone.

For the interval system $\mathcal{I} = \{[1,1],[2,2],\ldots,[n,n]\}$, abbreviate $D^2_{\mathcal{I}}(P)$ as DP, and call this the *horizontal double* of P. Thus the horizontal double of P is the poset obtained from P by replacing every $x \in P \setminus \{\hat{0},\hat{1}\}$ with two elements x_1, x_2, such that $\hat{0}$ and $\hat{1}$ remain the minimum and maximum elements of the partially ordered set, and $x_i < y_j$ if and only if $x < y$ in P. (In the Hasse diagram of P, every edge is replaced by $\bowtie$.)

Definition 4 A *half-Eulerian poset* is a graded partially ordered set whose horizontal double is Eulerian.

The flag f-vectors of P and its horizontal double are connected by the formula $f_S(DP) = 2^{|S|} f_S(P)$. Thus,

$$L_S(DP) = \ell_S(P). \tag{4}$$

Applying the definition of Eulerian to the horizontal double of a poset we get

Proposition 2.2 *A graded partially ordered set P is half-Eulerian if and only if for every interval $[x,y]$ of P,*

$$\sum_{i=1}^{\rho(x,y)-1} (-1)^{i-1} f_i([x,y]) = (1+(-1)^{\rho(x,y)})/2.$$

Corollary 1.3 can now be restated for half-Eulerian posets.

Proposition 2.3 *A graded partially ordered set is half-Eulerian if and only if $\ell_S([x,y]) = 0$ for every interval $[x,y] \subseteq P$ and every subset S of $[1,\rho(x,y)-1]$ that is not an even set.*

The flag vectors of the horizontal doubles of half-Eulerian posets span the Eulerian subspace, the subspace defined by the generalized Dehn-Sommerville equations. But the cones they determine may be different. Write $\mathcal{C}^{n+1}_{\mathcal{E}}$ for the closed cone of flag vectors of Eulerian posets of rank $n+1$, and $\mathcal{C}^{n+1}_{\mathcal{D}}$ for the closed cone of flag vectors of horizontal doubles of half-Eulerian posets. We do not know if the inclusion $\mathcal{C}^{n+1}_{\mathcal{D}} \subseteq \mathcal{C}^{n+1}_{\mathcal{E}}$ is actually equality.

For which interval systems $\mathcal{I}$ is $P(n,\mathcal{I},N)$ half-Eulerian?

Definition 5 An interval system $\mathcal{I}$ on $[1,n]$ is *even* if for every pair of intervals $I, J \in \mathcal{I}$ the intersection $I \cap J$ has an even number of elements. (In particular, $|I|$ must be even for every $I \in \mathcal{I}$.)

Our goal is to show that the posets $P(n, \mathcal{I}, N)$ are half-Eulerian if and only if $\mathcal{I}$ is an even interval system. For this we need to understand the intervals of the posets $P(n, \mathcal{I}, N)$.

Proposition 2.4 *The interval $[x, y] \subseteq P(n, \mathcal{I}, N)$ is isomorphic to $P(\rho(x, y) - 1, \mathcal{J}, N)$, where $\mathcal{J} = \{I - \rho(x) \,:\, I \in \mathcal{I},\, I \subseteq [\rho(x)+1, \rho(y)-1]\}$.*

Proof: Let $\rho(x) = r$ and $\rho(y) = s$. Construct $P(n, \mathcal{I}, N)$ by applying the operators D_I^N for all $I \in \mathcal{I}$ to a chain. Since the order of applying these operators is arbitrary, we may choose to apply first those for which I is not a subset of $[r+1, s-1]$. At this point for every x' of rank r and y' of rank s with $y' \geq x'$, the interval $[x', y']$ is isomorphic to a chain of rank $\rho(x', y')$. Applying the remaining operators D_I^N leaves the elements of rank at most r or of rank at least s unchanged, and has the same effect on $[x', y']$ as applying the operators D_{I-r}^N to a chain of rank $\rho(x', y')$. $\diamond$

The effect on the flag f-vector of applying the operator D_I^N to a poset of rank $n+1$ is given by the formula

$$f_S(D_I^N(P)) = \begin{cases} N f_S(P) & \text{if } I \cap S \neq \emptyset, \\ f_S(P) & \text{otherwise.} \end{cases} \tag{5}$$

This enables us to write an ℓ-vector formula.

Lemma 2.5 *For P a graded poset of rank $n+1$, $S \subseteq [1, n]$, and N a positive integer,*

$$\ell_S(D_I^N(P)) = N \ell_S(P) - (N-1) \sum_{T \cup I = S} \ell_T(P). \tag{6}$$

Proof: From the definition of ℓ_S and equation (5),

$$\begin{aligned} \ell_S(D_I^N(P)) &= (-1)^{n-|S|} \sum_{R \supseteq [1,n] \setminus S} (-1)^{|R|} f_R(D_I^N(P)) \\ &= (-1)^{n-|S|} \sum_{R \supseteq [1,n] \setminus S} (-1)^{|R|} N f_R(P) \\ &\quad - (-1)^{n-|S|} \sum_{\substack{R \supseteq [1,n] \setminus S \\ R \subseteq [1,n] \setminus I}} (-1)^{|R|} (N-1) f_R(P) \\ &= N \ell_S(P) - (-1)^{n-|S|} \sum_{\substack{R \supseteq [1,n] \setminus S \\ R \subseteq [1,n] \setminus I}} (-1)^{|R|} (N-1) f_R(P). \end{aligned}$$

By (3), the coefficient in $-(-1)^{n-|S|}\sum_{\substack{R\supseteq[1,n]\setminus S\\ R\subseteq[1,n]\setminus I}}(-1)^{|R|}(N-1)f_R(P)$ of $\ell_T(P)$ is

$$-(N-1)(-1)^{n-|S|}\sum_{\substack{R\supseteq[1,n]\setminus S\\ R\subseteq[1,n]\setminus(T\cup I)}}(-1)^{|R|},$$

which is an empty sum if $(T\cup I)$ is not contained in S, zero if $(T\cup I)$ is properly contained in S, and $-(N-1)(-1)^{n-|S|}(-1)^{|[1,n]\setminus S|}=-(N-1)$ if $(T\cup I)=S$. This gives the recursion of the lemma. $\diamond$

From this we can determine which of the posets $P(n,\mathcal{I},N)$ are half-Eulerian.

Proposition 2.6 *Let $\mathcal{I}$ be an interval system on $[1,n]$.*

1. *If $\mathcal{I}$ is an even system of intervals, then for all N the partially ordered set $P(n,\mathcal{I},N)$ is half-Eulerian.*
2. *If for some $N>1$, $P(n,\mathcal{I},N)$ is half-Eulerian, then $\mathcal{I}$ is an even system of intervals.*

Proof: Using Lemma 2.5 we can show by induction on $|\mathcal{I}|$ that for every N, $\ell_S^{n+1}(P(n,\mathcal{I},N))$ is zero unless S is the union of some intervals of $\mathcal{I}$. In particular, if $\mathcal{I}$ is an even system of intervals, then $\ell_S(P(n,\mathcal{I},N))=0$ whenever S is not an even set. The same observation holds for every interval $[x,y]\subseteq P(n,\mathcal{I},N)$ as well, since by Proposition 2.4 $[x,y]$ is isomorphic to $P(m,\mathcal{J},N)$ for some $m\le n$ and some even system of intervals $\mathcal{J}$. Therefore the conditions of Proposition 2.3 are satisfied by $P(n,\mathcal{I},N)$ for every N, if $\mathcal{I}$ is an even system of intervals.

Now assume $\mathcal{I}$ is a system of intervals that is not even. First consider the case where $\mathcal{I}$ contains an interval $I_m=[a,b]$ with $b-a$ even (hence I_m is odd). Let $\mathcal{J}=\{I_m-a+1\}=\{[1,b-a+1]\}$. For S nonempty, $f_S(P(b-a+1,\mathcal{J},N))=N$, so

$$\begin{aligned}\ell_{[1,b-a+1]}&(P(b-a+1,\mathcal{J},N))\\ &= \sum_{T\subseteq[1,b-a+1]}(-1)^{|T|}f_T(P(b-a+1,\mathcal{J},N))\\ &= 1+\sum_{\substack{T\subseteq[1,b-a+1]\\ T\neq\emptyset}}(-1)^{|T|}N=1-N.\end{aligned}$$

So $\ell_{[1,b-a+1]}(P(b-a+1,\mathcal{J},N))\neq 0$ for $N>1$. Fix $N>1$, and choose x and y in $P(n,\mathcal{I},N)$ with $\rho(x)=a-1$, $\rho(y)=b+1$, and $x\le y$. Then by

Proposition 2.4, $\ell_{[1,\rho(x,y)-1]}([x,y]) = \ell_{[1,b-a+1]}(P(b-a+1,\mathcal{J},N)) \neq 0$, with $|[1,b-a+1]|$ odd. So $P(n,\mathcal{I},N)$ is not half-Eulerian.

Now suppose $\mathcal{I}$ contains only even intervals, but some two intervals have an odd overlap. Let $I_p = [a,d]$ and $I_q = [c,b]$, where $a < c \leq d < b$ and $d-a$ and $b-c$ are odd, but $d-c$ is even. Then $b-a$ is also even. We show that we may assume no other interval of $\mathcal{I}$ is in the union $I_p \cup I_q$. Suppose $I_r = [e,f]$ is another interval of $\mathcal{I}$ with $[e,f] \subset [a,b]$ (and $f-e$ is odd). Since $\mathcal{I}$ is an antichain, $a < e < c \leq d < f < b$. If $e-a$ is even, then $|I_q \cap I_r| = |[c,f]| = f-c+1 = (f-e)+(e-a)-(d-a)+(d-c)+1$, which is odd, because it is the sum of three odds and two evens. If $e-a$ is odd, then $|I_p \cap I_r| = |[e,d]| = d-e+1 = (d-a)-(e-a)+1$, which is odd because it is the sum of three odds. Thus, if two intervals of $\mathcal{I}$ have odd intersection and their union contains a third interval of $\mathcal{I}$, then two intervals of $\mathcal{I}$ with smaller union have odd intersection.

So we may assume $I_p = [a,d]$ and $I_q = [c,b]$ have odd intersection, and their union $[a,b]$ contains no other interval of $\mathcal{I}$. Let $\mathcal{J} = \{I_p - a + 1, I_q - a + 1\} = \{[1, d-a+1], [c-a+1, b-a+1]\}$. Then

$$
\begin{aligned}
&f_S(P(b-a+1,\mathcal{J},N)) \\
&\qquad = \begin{cases} 1 & \text{if } S = \emptyset \\ N^2 & \text{if } S \cap (I_p - a + 1) \neq \emptyset \text{ and } S \cap (I_q - a + 1) \neq \emptyset \\ N & \text{otherwise.} \end{cases}
\end{aligned}
$$

So

$$
\begin{aligned}
&\ell_{[1,b-a+1]}(P(b-a+1,\mathcal{J},N)) \\
&\quad = \sum_{T \subseteq [1,b-a+1]} (-1)^{|T|} f_T(P(b-a+1,\mathcal{J},N)) \\
&\quad = \sum_{T \subseteq [1,b-a+1]} (-1)^{|T|} N^2 + \sum_{T \subseteq [1,c-a]} (-1)^{|T|}(N-N^2) \\
&\quad + \sum_{T \subseteq [d-a+2,b-a+1]} (-1)^{|T|}(N-N^2) + (1-2N+N^2) = (1-N)^2.
\end{aligned}
$$

So $\ell_{[1,b-a+1]}(P(b-a+1,\mathcal{J},N)) \neq 0$ for $N > 1$. Fix $N > 1$, and choose x and y in $P(n,\mathcal{I},N)$ with $\rho(x) = a-1$, $\rho(y) = b+1$, and $x \leq y$. Then by Proposition 2.4, $\ell_{[1,\rho(x,y)-1]}([x,y]) = \ell_{[1,b-a+1]}(P(b-a+1,\mathcal{J},N)) \neq 0$, with $|[1,b-a+1]|$ odd. So $P(n,\mathcal{I},N)$ is not half-Eulerian. $\diamond$

As will be seen later, even interval systems give rise to extreme rays of the cone of flag vectors of Eulerian posets. It is of interest, therefore, to count them.

Proposition 2.7 *The number of even interval systems on $[1,n]$ is $\binom{n}{\lfloor n/2 \rfloor}$.*

Proof: We define a one-to-one correspondence between even interval systems on $[1,n]$ and sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \{-1,1\}^n$ satisfying $\sum_i \lambda_i = 0$ if n is even and $\sum_i \lambda_i = 1$ if n is odd. Clearly there are $\binom{n}{\lfloor n/2 \rfloor}$ such sequences.

For $\mathcal{I}$ an even interval system, define $\lambda(\mathcal{I}) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \{-1,1\}^n$, where $\lambda_i = (-1)^i$ if i is an endpoint of an interval of $\mathcal{I}$, and $\lambda_i = (-1)^{i-1}$ otherwise. (Note that for an even interval system, no number can be an endpoint of more than one interval.) For $\mathcal{I}$ an even interval system, summing $(-1)^i$ over the endpoints of intervals gives 0. So

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i &= \sum_{i=1}^{n} (-1)^{i-1} + \sum_{\substack{i \text{ endpoint} \\ \text{of interval}}} 2(-1)^i \\ &= \sum_{i=1}^{n} (-1)^{i-1} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} . \end{aligned}$$

On the other hand, given a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \{-1,1\}^n$ satisfying $\sum_i \lambda_i = 0$ if n is even and $\sum_i \lambda_i = 1$ if n is odd, construct an even interval system as follows. Let $s_1 < s_2 < \cdots < s_k$ be the sequence of indices s for which $\lambda_s = (-1)^s$. Then $\sum_{i=1}^{n} (-1)^{i-1} = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} (-1)^{i-1} + \sum_{j=1}^{k} 2(-1)^{s_j}$, so $\sum_{j=1}^{k} (-1)^{s_j} = 0$. Thus the sequence of s_j's contains the same number of even numbers as odd. Construct an interval system $\mathcal{I} = \{[a_1,b_1],[a_2,b_2],\ldots,[a_m,b_m]\}$ $(2m = k)$ recursively as follows. Let $a_1 = s_1$ and let $b_1 = s_j$ where j is the least index such that s_1 and s_j are of opposite parity. Then $\mathcal{I} = [a_1,b_1] \cup \mathcal{I}'$, where $\mathcal{I}'$ is the interval system associated with $s_2 < s_3 < s_4 < \cdots < s_k$ with $b_1 = s_j$ removed. Clearly $[a_1,b_1]$ is of even length. If $[a_1,b_1] \cap [a_i,b_i] \neq \emptyset$ for some interval $[a_i,b_i]$ of $\mathcal{I}'$, then $a_i < b_1$, so by the choice of b_1, a_i has the same parity as a_1. Thus $[a_1,b_1] \cap [a_i,b_i] = [a_i,b_1]$ is of even length. Furthermore, b_i and b_1 are of the same parity, since a_i and a_1 are, so again by the choice of b_1, $b_i > b_1$. So the interval $[a_i,b_i]$ is not contained in the interval $[a_1,b_1]$. The interval system $\{[a_m,b_m]\}$, is even, so by induction $\mathcal{I}$ is an even interval system.

These constructions are inverses, giving the desired bijection. ◇

Recall that Billera and Hetyei ([8]) found extremes of the cone of flag vectors of graded posets as limits of the normalized flag vectors of the posets $P(n,\mathcal{I},N)$. The next proposition follows easily by induction from Lemma 2.5.

Proposition 2.8 *Let $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$ be a system of $k \geq 0$ intervals on $[1, n]$. Then*

$$\begin{aligned}\lim_{N \longrightarrow \infty} & \frac{1}{N^k} \ell_S \left(P(n, \mathcal{I}, N)\right) \\ & = \sum_{j=0}^{k} (-1)^j \left| \left\{ 1 \leq i_1 < \cdots < i_j \leq k \,:\, I_{i_1} \cup \cdots \cup I_{i_j} = S \right\} \right|.\end{aligned}$$

Write $f_S(P(n, \mathcal{I})) = \lim_{N \to \infty} f_S(P(n, \mathcal{I}, N))/N^{|\mathcal{I}|}$. The vector these numbers form (as S ranges over all subsets of $[1, n]$) is not the flag f-vector of an actual poset, but it is in the closed cone of flag f-vectors of all graded posets. We call the symbol $P(n, \mathcal{I})$ a "limit poset" and refer to the flag vector of the limit poset. If $\mathcal{I}$ is an even interval system, then $(f_S(P(n, \mathcal{I})) \,:\, S \subseteq [1, n])$ is in the closed cone of flag vectors of half-Eulerian posets. To get Eulerian posets the horizontal double operator is applied to $P(n, \mathcal{I}, N)$. The vector $(f_S(DP(n, \mathcal{I})) \,:\, S \subseteq [1, n])$ is defined as a limit of the resulting normalized flag f-vectors, and satisfies $f_S(DP(n, \mathcal{I})) = 2^{|S|} f_S(P(n, \mathcal{I}))$. It lies in the cone $\mathcal{C}_{\mathcal{D}}^{n+1}$ of flag vectors of doubles of half-Eulerian posets, a subcone of the Eulerian cone.

Recall (equation (4)) that the ℓ-vector of a poset P equals the L-vector of its horizontal double DP. The same holds after passing to the limit posets. Thus, Proposition 2.8 gives

$$L_S(DP(n, \mathcal{I})) = \sum_{j=0}^{k} (-1)^j \left| \left\{ 1 \leq i_1 < \cdots < i_j \leq k : I_{i_1} \cup \cdots \cup I_{i_j} = S \right\} \right|,$$

where $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$.

We look at the associated *cd*-indices of the "doubled limit posets." Think of a word in c and d as a string with each c occupying one position and each d occupying two positions. The *weight* of a *cd*-word w is then the number of positions of the string. Associated to each *cd*-word w is the even set $S(w)$ consisting of the positions occupied by the d's.

Proposition 2.9 *For each cd-word w with k d's and weight n, there exists an even interval system $\mathcal{I}_w$ for which the cd-index of $DP(n, \mathcal{I}_w)$ is $2^k w$.*

Proof: Fix a *cd*-word w with k d's and weight n. Write the elements of $S(w)$ in increasing order as i_1, $i_1 + 1$, i_2, $i_2 + 1$, $\ldots$, i_k, $i_k + 1$, and let $\mathcal{I}_w$ be the interval system $\{[i_1, i_1 + 1], [i_2, i_2 + 1], \ldots, [i_k, i_k + 1]\}$. Let $\Phi = 2^k w$. Rewrite the *cd*-polynomial Φ as a *ce*-polynomial. Recall from Sections 1.1

and 1.2 that $c = a + b$, $d = ab + ba$, and $e = a - b$, so $d = (cc - ee)/2$. Thus, Φ is rewritten as a sum of 2^k terms. Each is the result of replacing some subset of the d's by cc, and the rest by ee; the coefficient is ± 1, depending on whether the number of d's replaced by ee is even or odd. Thus

$$2^k w = \sum_{J \subseteq [1,k]} (-1)^{|J|} w_J,$$

where $w_J = w_1 w_2 \cdots w_n$, with $w_{i_j} = w_{i_j+1} = e$ if $j \in J$ and the remaining w_i's are c. By the L-vector version of Proposition 2.8, this is precisely the ce-index of $DP(n, \mathcal{I}_w)$. $\diamond$

In [20] Stanley first found for each cd-word w a sequence of Eulerian posets whose normalized cd-indices converge to w. Our limit posets are closely related to Stanley's, but this particular construction highlights the important link between the half-Eulerian and Eulerian cones.

Before turning to inequalities satisfied by the flag vectors of Eulerian posets, we consider the question of whether the two cones $\mathcal{C}_{\mathcal{D}}^{n+1}$ and $\mathcal{C}_{\mathcal{E}}^{n+1}$ are equal. For low ranks the two cones are the same, as seen below. We know of no example in any rank of an Eulerian poset whose flag vector is not contained in the cone $\mathcal{C}_{\mathcal{D}}^{n+1}$ of doubled half-Eulerian posets. To look for such an example we turn to the best known examples of Eulerian posets, the face lattices of polytopes. In [20] Stanley proved the nonnegativity of the cd-index for "S-shellable regular CW-spheres", a class of Eulerian posets that includes all polytopes. By a result of Billera, Ehrenborg, and Readdy ([7]), the lattice of regions of any oriented matroid also has a nonnegative cd-index. (Some entries in the cd-index are nonnegative for all Eulerian posets; see [3] for details.) Proposition 2.9 implies that nonnegative cd-indices (and the associated flag vectors) are in the cone generated by the cd-indices (flag vectors) of the doubles of limit posets associated with even interval systems.

Corollary 2.10 *$\mathcal{C}_{\mathcal{D}}^{n+1}$ contains the flag vectors of all Eulerian posets with nonnegative cd-indices. This includes the face lattices of polytopes and the lattices of regions of oriented matroids.*

Conjecture 2.11 *The closed cone $\mathcal{C}_{\mathcal{E}}^{n+1}$ of flag vectors of Eulerian posets is the same as the closed cone $\mathcal{C}_{\mathcal{D}}^{n+1}$ of flag vectors of horizontal doubles of half-Eulerian posets.*

3 Inequalities

Throughout this section we use the following notation.

Definition 6 The *interval system* $\mathcal{I}[S]$ *of a set* $S \subseteq [1,n]$ is the family of intervals $\mathcal{I}[S] = \{[a_1,b_1],\ldots,[a_k,b_k]\}$, where $S = [a_1,b_1] \cup \cdots \cup [a_k,b_k]$ and $b_{i-1} < a_i - 1$ for $i \geq 2$. In other words, $\mathcal{I}[S]$ is the collection of the maximal intervals contained in S.

Note that S is an even set if and only if $\mathcal{I}[S]$ is an even interval system.

The following flag vector forms can be proved nonnegative by writing them as convolutions of basic nonnegative forms ([10, 15]). (See Appendix B.) The issue of whether they give all linear inequalities on flag vectors of Eulerian posets was raised by Billera and Liu (see the discussion after Proposition 1.3 in [10]). We give here a simple direct argument for their nonnegativity that avoids convolutions.

Proposition 3.1 (Inequality Lemma) *Let T and V be subsets of $[1,n]$ with $T \subseteq V$, such that for every $I \in \mathcal{I}[V]$, $|I \cap T| \leq 1$. Write $S = [1,n] \setminus V$. For P any rank $n+1$ Eulerian poset,*

$$\sum_{R \subseteq T} (-2)^{|T \setminus R|} f_{S \cup R}(P) \geq 0.$$

Equivalently,

$$(-1)^{|T|} \sum_{T \subseteq Q \subseteq V} L_Q(P) \geq 0.$$

Proof: The idea is that since no two elements of T are in the same gap of S, elements with ranks in T can be inserted independently in chains with rank set S. For C an S-chain (i.e., a chain with rank set S) and $t \in T$, let $n_t(C)$ be the number of rank t elements $x \in P$ such that $C \cup \{x\}$ is a chain of P. Since every interval of an Eulerian poset is Eulerian, $n_t(C) \geq 2$ for all C and t. So

$$\begin{aligned}
\sum_{R \subseteq T} (-2)^{|T \setminus R|} f_{S \cup R}(P) &= \sum_{R \subseteq T} (-2)^{|T \setminus R|} \sum_{C \text{ an } S\text{-chain}} \prod_{t \in R} n_t(C) \\
&= \sum_{C \text{ an } S\text{-chain}} \sum_{R \subseteq T} (-2)^{|T \setminus R|} \prod_{t \in R} n_t(C) \\
&= \sum_{C \text{ an } S\text{-chain}} \prod_{t \in T} (n_t(C) - 2) \geq 0.
\end{aligned}$$

So the flag vector inequality is proved. The second inequality is simply the translation into L-vector form. $\diamond$

Here are some new inequalities.

Theorem 3.2 *Let $1 \le i < j < k \le n$. For P any rank $n+1$ Eulerian poset,*

$$f_{ik}(P) - 2f_i(P) - 2f_k(P) + 2f_j(P) \ge 0.$$

Proof: First order the rank j elements of P in the following way. Choose any order, $G_1, G_2, \ldots, G_m$ for the components of the Hasse diagram of the rank-selected poset $P_{\{i,j,k\}}$. For each rank j element y of P, identify the component containing y by $y \in G_{g(y)}$. Order the rank j elements of P in any way consistent with the ordering of components. That is, choose an order $y_1, y_2, \ldots, y_r$ such that $y_s < y_t$ implies $g(y_s) \le g(y_t)$.

A rank i element x *belongs* to y_q if q is the least index such that $x < y_q$ in P. Write I_q for the number of rank i elements belonging to y_q, and I'_q for the number of rank i elements x such that $x < y_q$, but x does not belong to y_q. Similarly, a rank k element z *belongs* to y_q if q is the least index such that $y_q < z$ in P. Write K_q for the number of rank k elements belonging to y_q, and K'_q for the number of rank k elements z such that $y_q < z$, but z does not belong to y_q. Note that $I_q + I'_q \ge 2$ and $K_q + K'_q \ge 2$, since P is Eulerian. A flag $x < z$ *belongs* to y_q if $x < y_q < z$ and q is the least index such that either $x < y_q$ or $y_q < z$.

Let $F = f_{ik}(P) - 2f_i(P) - 2f_k(P) + 2f_j(P)$. Let F_q be the contribution to F by elements and flags belonging to y_q. Thus,

$$F_q = I_qK_q + I'_qK_q + I_qK'_q - 2I_q - 2K_q + 2.$$

If $I'_q \ge 2$, then $F_q = I_q(K_q + K'_q - 2) + (I'_q - 2)K_q + 2 \ge 2$.
If $I'_q = K'_q = 0$, then $F_q = (I_q - 2)(K_q - 2) - 2 \ge -2$.
In all other cases it is easy to check that $F_q \ge 0$.

Suppose that the rank j elements in component G_ℓ are $y_s, y_{s+1}, \ldots, y_t$. Then $I'_s = K'_s = 0$, so $F_s \ge -2$. Furthermore, $I_t = K_t = 0$, because any rank i element x related to y_t must also be related to at least one other rank j element, and it is in the same component. That rank j element has index less than t, so x does not belong to y_t. This in turn implies $I'_t \ge 2$, so $F_t \ge 2$. For all q, $s < q < t$, either $I'_q > 0$ or $K'_q > 0$, by the connectivity of the component, so $F_q \ge 0$. Thus $\sum_{q=s}^{t} F_q \ge 0$. This is true for each component G_ℓ, so $F = \sum_{q=1}^{r} F_q \ge 0$. $\diamond$

These inequalities can be used to generate others by convolution (see Appendix B.)

Evaluating the flag vector inequalities of Proposition 3.1 for the horizontal double DP of a half-Eulerian poset P gives the inequalities, for S and T

satisfying the hypotheses of Proposition 3.1,

$$\sum_{R \subseteq T} (-1)^{|T \setminus R|} f_{S \cup R}(P) \geq 0. \tag{7}$$

These inequalities are valid not just for half-Eulerian posets but for *all* graded posets. The proof of Proposition 3.1 uses only the fact that in every open interval of an Eulerian poset there are at least two elements of each rank. If the proof is rewritten using the assumption that in every open interval there is at least one element of each rank, the inequalities (7) are proved for all graded posets.

Similarly, the flag vector inequalities of Theorem 3.2 give inequalities for half-Eulerian posets,

$$f_{ik}(P) - f_i(P) - f_k(P) + f_j(P) \geq 0.$$

The proof of Theorem 3.2 can be modified in the same way to show these inequalities are valid for all graded posets. The first instance of this class of inequalities was found by Billera and Liu ([10]).

We conjecture that all inequalities valid for half-Eulerian posets come from inequalities valid for all graded posets. Inequalities for half-Eulerian posets are to be interpreted as conditions in the subspace of $\mathbf{R}^{2^n}$ spanned by flag vectors of half-Eulerian posets, but we are describing them in $\mathbf{R}^{2^n}$. Giving inequalities using linear forms in the flag numbers f_S over $\mathbf{R}^{2^n}$, the statement is as follows.

Conjecture 3.3 *Every linear form that is nonnegative for the flag vectors of all half-Eulerian posets is the sum of a linear form that is nonnegative for all graded posets and a linear form that is zero for all half-Eulerian posets.*

4 Extreme Rays and Facets of the Cone

We have described some points in the Eulerian cone $\mathcal{C}_{\mathcal{E}}^{n+1}$ and some inequalities satisfied by all points in the cone. We turn now to identifying which of these give extreme rays and facets.

If $\mathcal{I}$ is an even interval system, then $(f_S(P(n, \mathcal{I})) : S \subseteq [1, n])$ is on an extreme ray in the closed cone of flag vectors of all graded posets, and is in the subcone of flag f-vectors of half-Eulerian posets. Therefore it is on an extreme ray of the subcone. By Proposition 2.7 this gives $\binom{n}{\lfloor n/2 \rfloor}$ extreme rays for the rank $n+1$ cone.

Proposition 4.1 *For every even interval system $\mathcal{I}$, the flag vector of the limit poset $P(n,\mathcal{I})$ generates an extreme ray of the cone of flag vectors of half-Eulerian posets.*

What does this say about the extreme rays of the cone of flag vectors of Eulerian posets? For every even interval system $\mathcal{I}$, the flag vector of $DP(n,\mathcal{I})$ lies on an extreme ray of the subcone $\mathcal{C}_{\mathcal{D}}^{n+1}$, but we cannot conclude directly that it lies on an extreme ray of the cone $\mathcal{C}_{\mathcal{E}}^{n+1}$. A separate proof is needed.

For the following proofs, we use the computation of $\ell_Q(P(n,\mathcal{I}))$ (and $L_Q(DP(n,\mathcal{I}))$) from the decompositions of Q as the union of intervals of $\mathcal{I}$ (Proposition 2.8).

Theorem 4.2 *For every even interval system $\mathcal{I}$, the flag vector of the doubled limit poset $DP(n,\mathcal{I})$ generates an extreme ray of the cone of flag vectors of Eulerian posets.*

Proof: We work in the closed cone of L-vectors of Eulerian posets. The cone of L-vectors of Eulerian posets is contained in the subspace of $\mathbf{R}^{2^n}$ determined by the equations $L_S = 0$ for S not an even set. To prove that the L-vector of $DP(n,\mathcal{I})$ generates an extreme ray, we show that it lies on linearly independent supporting hyperplanes, one for each nonempty even set V in $[1,n]$. Fix an even interval system $\mathcal{I}$. For each nonempty even set $V \subseteq [1,n]$, we find a set T such that T and V satisfy the hypothesis of Proposition 3.1 and $\sum_{T\subseteq Q\subseteq V} L_Q(DP(n,\mathcal{I})) = 0$.

Case 1. Suppose V is the union of some intervals in $\mathcal{I}$. Let I_1, I_2, $\ldots$, I_k be all the intervals of $\mathcal{I}$ contained in V. Set $T = \emptyset$. Then for each subset $J \subseteq [1,k]$, the corresponding union of intervals contributes $(-1)^{|J|}$ to $L_Q(DP(n,\mathcal{I}))$, for $Q = \cup_{j\in J} I_j$. Thus $\sum_{T\subseteq Q\subseteq V} L_Q(DP(n,\mathcal{I})) = \sum_{J\subseteq[1,k]}(-1)^{|J|} = 0$.

Case 2. If V is not the union of some intervals in $\mathcal{I}$, let W be the union of all those intervals of $\mathcal{I}$ contained in V. Choose $t \in V\setminus W$, and set $T = \{t\}$. For $Q \subseteq V$, $L_Q(DP(n,\mathcal{I})) = 0$ unless $Q \subseteq W$. But if $Q \subseteq W$ then t cannot be in Q. So $\sum_{\{t\}\subseteq Q\subseteq V} L_Q(DP(n,\mathcal{I})) = 0$.

Now $\sum_{T\subseteq Q\subseteq V} L_Q(P) = 0$ determines a supporting hyperplane of the closed cone of L-vectors of Eulerian posets, because the inequality of Proposition 3.1 is valid, and the poset $DP(n,\mathcal{I})$ lies on the hyperplane. The hyperplane equations each involve a distinct maximal set V, which is even, so they are linearly independent on the subspace determined by the equations $L_S = 0$ for S not an even set. So the doubled limit poset $DP(n,\mathcal{I})$ is on an extreme ray of the cone. $\diamond$

Note how far we are, however, from a complete description of the extreme rays.

Conjecture 4.3 *For every positive integer n, the closed cone of flag f-vectors of Eulerian posets of rank $n+1$ is finitely generated.*

Lemma 4.4 (Facet Lemma) *Assume $\sum_{Q\subseteq[1,n]} a_Q L_Q(P) \geq 0$ for all Eulerian posets P of rank $n+1$. Let $M \subseteq [1,n]$ be a fixed even set. Suppose for all even sets $R \subseteq [1,n]$, $R \neq M$, there exists an interval system $\mathcal{I}(R)$ consisting of disjoint even intervals whose union is R and such that $\sum_{Q\subseteq[1,n]} a_Q \ell_Q(P(n,\mathcal{I}(R))) = 0$. Then $\sum_{Q\subseteq[1,n]} a_Q L_Q(P) = 0$ determines a facet of the closed cone of L-vectors of Eulerian posets.*

(Note that $\mathcal{I}(R)$ need not be $\mathcal{I}[R]$.)
Proof: The dimension of the cone $\mathcal{C}_{\mathcal{E}}^{n+1}$ equals the number of even subsets (a Fibonacci number). So it suffices to show that the vectors $(\ell_Q(P(n,\mathcal{I}(R))))$ $= (L_Q(DP(n,\mathcal{I}(R))))$ are linearly independent. To see this, note that for every set Q not contained in R, $\ell_Q(P(n,\mathcal{I}(R))) = 0$. By the disjointness of the intervals in $\mathcal{I}(R)$, there is a unique way to write R as the union of intervals in $\mathcal{I}(R)$. So by Proposition 2.8, $(\ell_R(P(n,\mathcal{I}(R)))) = (-1)^{|\mathcal{I}(R)|}$. Thus, R is the unique maximal set Q for which $(\ell_Q(P(n,\mathcal{I}(R)))) \neq 0$. So the L-vectors of the posets $DP(n,\mathcal{I}(R))$, as R ranges over sets different from M, are linearly independent. $\diamond$

Proposition 4.5 *The inequality $\sum_{Q\subseteq[1,n]} L_Q(P) \geq 0$ (or, equivalently, $f_\emptyset(P) \geq 0$) determines a facet of the closed cone of L-vectors of Eulerian posets of rank $n+1$.*

Proof: Apply the Facet Lemma 4.4 with $M = \emptyset$. For a nonempty even set R, the interval system $\mathcal{I}[R]$ of R is nonempty, so $\sum_{Q\subseteq[1,n]} \ell_Q(P(n,\mathcal{I}[R])) = \sum_{\mathcal{J}\subseteq\mathcal{I}[R]} (-1)^{|\mathcal{J}|} = 0$. $\diamond$

Theorem 4.6 *Let V be a subset of $[1,n]$ such that every $I \in \mathcal{I}[V]$ has cardinality at least 2, and every $I \in \mathcal{I}[[0,n+1] \setminus V]$ has cardinality at most 3. Assume that M is a subset of V such that every $[a,b] \in \mathcal{I}[V]$ satisfies the following:*

(i) $M \cap [a,b] = \emptyset$, $[a,a+1]$, *or* $[b-1,b]$.

(ii) *If* $a \notin M$ *then* $a-2 \in \{-1\} \cup M$.

(iii) If $b \notin M$ then $b+2 \in \{n+2\} \cup M$.

Then

$$(-1)^{|M|/2} \sum_{M \subseteq Q \subseteq V} L_Q(P) \geq 0 \tag{8}$$

determines a facet of $\mathcal{C}_{\mathcal{E}}^{n+1}$. Furthermore, if we strengthen (i) by also requiring $M \cap [a, a+2] = \emptyset$ for every $[a, a+2] \in \mathcal{I}[V]$, then distinct pairs (M, V) give distinct facets.

Proof: If $M = \emptyset$, then conditions (ii) and (iii) force $V = [1, n]$ (or $V = \emptyset$ if $n \leq 1$). The resulting inequality, $\sum_{Q \subseteq [1,n]} L_Q(P) \geq 0$, gives a facet, as shown in Proposition 4.5. Now assume that $M \neq \emptyset$.

Step 1 is to prove that inequality (8) holds for all Eulerian posets. Note that $\mathcal{I}[M]$ is a nonempty collection of intervals of length two. From each such interval choose one endpoint adjacent to an element of $[0, n+1] \setminus V$. Let T be the set of these chosen elements. The Inequality Lemma 3.1 applies to these T and V because each interval of V contains at most one interval of $\mathcal{I}[M]$, and hence at most one element of T. The resulting inequality is $(-1)^{|T|} \sum_{T \subseteq Q \subseteq V} L_Q \geq 0$. Now $L_Q(P) = 0$ for all P if $\mathcal{I}[Q]$ contains an odd interval. So we can restrict the sum to even sets Q. Since Q must be contained in V, such a Q must contain the intervals of M. Thus, $(-1)^{|M|/2} \sum_{M \subseteq Q \subseteq V} L_Q(P) \geq 0$.

Step 2 is to prove that if $I \subseteq [1, n]$ is an interval of cardinality at least 2 and I contains an element i not in V, then I contains an element adjacent to an interval of M. If an interval from $\mathcal{I}[V]$ ends at $i-1$, then either $i-1 \in M$ or $i+1 \in M$ by (iii) (since $i+1 < n+2$). Similarly, if an interval from $\mathcal{I}[V]$ begins at $i+1$, then either $i-1 \in M$ or $i+1 \in M$. So assume no interval from $\mathcal{I}[V]$ begins at $i-1$ or ends at $i+1$. The hypothesis of the theorem states that every interval from $\mathcal{I}[[0, n+1] \setminus V]$ has cardinality at most three. Thus the interval $[i-1, i+1]$ belongs to $\mathcal{I}[[0, n+1] \setminus V]$. Hence $i-2 \in \{-1\} \cup V$ and $i+2 \in \{n+2\} \cup V$. If $i-2 = -1$ then $I \supseteq [i, i+1] = [1, 2]$, condition (ii) applied to $a = 3$ yields $3 \in M$, and $2 \in I$ is adjacent to 3. The case when $i+2 = n+2$ is dealt with similarly. Finally, if $i-2$ and $i+2$ are both endpoints of intervals from $\mathcal{I}[V]$, then, since $i \notin M \cup \{-1, n+2\}$, condition (ii) applied to $a = i+2$ and condition (iii) applied to $b = i-2$ yield $i+2 \in M$ and $i-2 \in M$. Either $i-1$ or $i+1$ belongs to I and each of them is adjacent to an element of M.

Recall that for $\mathcal{I}$ an even interval system, the vector $(\ell_Q(P(n, \mathcal{I})) : Q \subseteq [1, n])$ is in the closed cone of ℓ-vectors of half-Eulerian posets. Step

3 is to show that for each even set $R \neq M$, there exists an even interval system $\mathcal{I}$ with $\cup_{i \in \mathcal{I}} I = R$ such that $(-1)^{|M|/2} \sum_{M \subseteq Q \subseteq V} \ell_Q(P(n, \mathcal{I})) = 0$.

Let R be an even set not equal to M. If $M \not\subseteq R$, then for every Q containing M, $\ell_Q(P(n, \mathcal{I}[R])) = 0$. Now suppose $M \subseteq R$, but $R \not\subseteq V$. Let I be an interval of $\mathcal{I}[R]$ such that $I \not\subseteq V$. Then I contains an element adjacent to an interval of M. Since $M \subseteq R$ and I is a maximal interval in R, $I \cap M \neq \emptyset$. Thus every union of intervals of $\mathcal{I}[R]$ containing M must contain I and thus an element not in V. So $\sum_{M \subseteq Q \subseteq V} \ell_Q(P(n, \mathcal{I}[R])) = 0$, because all terms are zero.

Finally, suppose $M \subseteq R \subseteq V$ and $R \neq M$. Let $\mathcal{I}$ be the interval system of R consisting only of intervals of length 2. Then every interval of M is in $\mathcal{I}$. This is because every interval of M is of length 2, with at least one of its endpoints adjacent to an element not in V. So $\sum_{M \subseteq Q \subseteq V} \ell_Q(P(n, \mathcal{I})) = \sum_{\mathcal{I}[M] \subseteq \mathcal{J} \subseteq \mathcal{I}} (-1)^{|\mathcal{J}|} = 0$, since $R \neq M$ implies $\mathcal{I} \neq \mathcal{I}[M]$.

By the Facet Lemma 4.4, the inequality $(-1)^{|M|/2} \sum_{M \subseteq Q \subseteq V} L_Q(P) \geq 0$ gives a facet of $\mathcal{C}_{\mathcal{E}}^{n+1}$.

Now we show that under the added condition $M \cap [a, a+2] = \emptyset$ for every $[a, a+2] \in \mathcal{I}[V]$, the facets obtained are distinct.

Note that two (M, V) pairs can give the same inequality only if they have the same M, because L_M is included in the linear form for (M, V), and M is the minimal (by set inclusion) set for which L_M is in the form. Now for fixed M, we show that (M, V_1) and (M, V_2) give distinct linear inequalities when $V_1 \neq V_2$. Since the sets V_1 and V_2 are different, there is an interval $[a, b]$ such that $[a, b]$ occurs in exactly one of $\mathcal{I}[V_1]$ or $\mathcal{I}[V_2]$. Let $[a, b]$ be a maximal interval with this property. Without loss of generality assume $[a, b] \in \mathcal{I}[V_1]$. Then $[a, b]$ is contained in no interval of $\mathcal{I}[V_2]$.

Case 1. $M \cap [a, b] = \emptyset$. Then for every i, $a \leq i \leq b-1$, the term $L_{[i,i+1] \cup M}$ occurs in the inequality for (M, V_1). At least one of these terms does not occur in the inequality for (M, V_2), because $[a, b] \not\subseteq V_2$.

Case 2. $M \cap [a, b] = [a, a+1]$. Since $M \subseteq V_2$ and $[a, b] \not\subseteq V_2$, $b > a+1$. By the strengthened hypothesis on M, $b \geq a+3$. Then for every i, $a+2 \leq i \leq b-1$, the term $L_{[i,i+1] \cup M}$ occurs in the inequality for (M, V_1). At least one of these terms does not occur in the inequality for (M, V_2), because $[a, b] \not\subseteq V_2$.

Case 3. $M \cap [a, b] = [b-1, b]$. The proof is similar to Case 2.

Thus, with the condition $M \cap [a, a+2] = \emptyset$ for every $[a, a+2] \in \mathcal{I}[V]$, the facets given by the theorem are all distinct. $\diamond$

Theorem 4.6 may be restated and interpreted in terms of the convolution of chain operators. We refer the interested reader to Appendix B for that approach.

With the aid of PORTA ([14]), we calculated the Eulerian cone for rank at most 7. Our input files, the output generated by PORTA, and a LaTeX file identifying the valid inequalities obtained may be found at [5]. It turns out that Theorems 4.2 and 4.6 give all the extremes and facets of the cone for rank at most 6. This fails at rank 7; there the facet inequalities all come from Proposition 3.1 and Theorem 3.2, but new poset constructions are needed for the extreme rays.

Theorem 4.7 *For rank $n+1 \leq 6$, the closed cone $\mathcal{C}_{\mathcal{E}}^{n+1}$ of flag vectors of Eulerian posets is finitely generated. It has $\binom{n}{\lfloor n/2 \rfloor}$ extreme rays, all generated by the flag vectors of the limit posets $DP(n,\mathcal{I})$ for $\mathcal{I}$ even interval systems on $[1,n]$. It has $\binom{n}{\lfloor n/2 \rfloor}$ facets, all given by Proposition 4.5 and Theorem 4.6.*

Theorem 4.8 (*i*) *The cone $\mathcal{C}_{\mathcal{E}}^{7}$ is finitely generated, with 24 extreme rays. Twenty of the extreme rays are generated by the flag vectors of the limit posets $DP(n,\mathcal{I})$ for $\mathcal{I}$ even interval systems on $[1,6]$.*

(*ii*) *The cone $\mathcal{C}_{\mathcal{E}}^{7}$ has 23 facets. Fifteen of the facets are given by the inequalities of Theorem 4.6. Four additional facets come from the Inequality Lemma 3.1. The remaining four come from Theorem 3.2.*

The four special extreme rays of the rank 7 Eulerian cone have corresponding rays in the half-Eulerian cone. The generators for the half-Eulerian cone are all obtained by adding the flag vectors of limit posets associated with noneven interval systems. The summands do not satisfy the conditions of Proposition 2.3 for half-Eulerian posets, but the sum does. The calculations are easily done in terms of the ℓ-vector, using Proposition 2.8. Specific sequences of half-Eulerian posets have been constructed whose flag vectors converge to these four extremes. The half-Eulerian posets are obtained by "gluing together" posets for each summand. These are then converted to Eulerian posets by the horizontal doubling operation. Below are the sums of limit posets used. Descriptions of the half-Eulerian posets are found in Appendix A.

Extreme 1: $P(6,\{[1,2],[2,6]\}+\{[2,5],[5,6]\})$
Extreme 2: $P(6,\{[1,3],[3,4],[4,6]\}+\{[1,2],[2,3]\}+\{[4,5],[5,6]\})$
Extreme 3: $P(6,\{[1,2],[3,4],[4,5]\}+\{[3,5],[5,6]\}+\{[1,2],[2,5]\})$
Extreme 4: $P(6,\{[1,2],[2,4]\}+\{[2,5],[5,6]\}+\{[2,3],[3,4],[5,6]\})$

Note that for rank at most 7, the two cones $\mathcal{C}_{\mathcal{D}}^{n+1}$ and $\mathcal{C}_{\mathcal{E}}^{n+1}$ are equal, because the generators of extreme rays specified in Theorems 4.7 and 4.8 are horizontal doubles of half-Eulerian limit posets.

Perhaps all the extreme rays of the half-Eulerian cone (if not the Eulerian cone) can be obtained by gluing together Billera-Hetyei limit posets.

A complete description of the closed cone of flag vectors of Eulerian posets remains open, and, as mentioned before, the cone is not even known to be finitely generated. We do not know if convolutions of the inequalities of Proposition 3.1 and Theorem 3.2 completely determine the cone. A better understanding of the construction of extreme rays as sums of Billera-Hetyei limit posets would be valuable.

The study of Eulerian posets is motivated in part by questions about convex polytopes. Is the cone of flag vectors of all Eulerian posets the same as or close to the cone of flag vectors of polytopes? The answer is no. The inequalities of Proposition 3.1 can be strengthened considerably for polytopes. The proof of Proposition 3.1 uses only the fact that in an Eulerian poset each interval has at least two elements of each rank. For convex polytopes, each interval is at least the size of a Boolean algebra of the same rank. Thus, for example, where Proposition 3.1 gives that $f_{1479}(P) - 2f_{179}(P) \geq 0$ for Eulerian posets, for convex polytopes the inequality $f_{1479}(P) - 20f_{179}(P) \geq 0$ holds, because the rank 6 Boolean algebra has $\binom{6}{3} = 20$ elements of rank 3. For ranks 4 through 7, we have verified that none of the extreme rays of the Eulerian cone is in the closed cone of flag vectors of convex polytopes.

Appendix A Some half-Eulerian limit posets of rank 7

Here are the constructions of half-Eulerian posets whose doubles give Extremes 1, 2 and 3 of $\mathcal{C}_{\mathcal{E}}^{7}$. Extreme 4 is the dual of Extreme 3.

In the following, C^7 denotes a chain of rank 7.

A.1 $P\left(6, \{[1,2],[2,6]\} + \{[2,5],[5,6]\}\right)$

Take $D_{[1,2]}^{N} D_{[2,6]}^{N}\left(C^7\right)$ and $D_{[1,5]}^{N} D_{[5,6]}^{N}\left(C^7\right)$. Identify the elements of both posets at rank 1 and at rank 6. Figure 1 represents the resulting poset for $N = 2$.

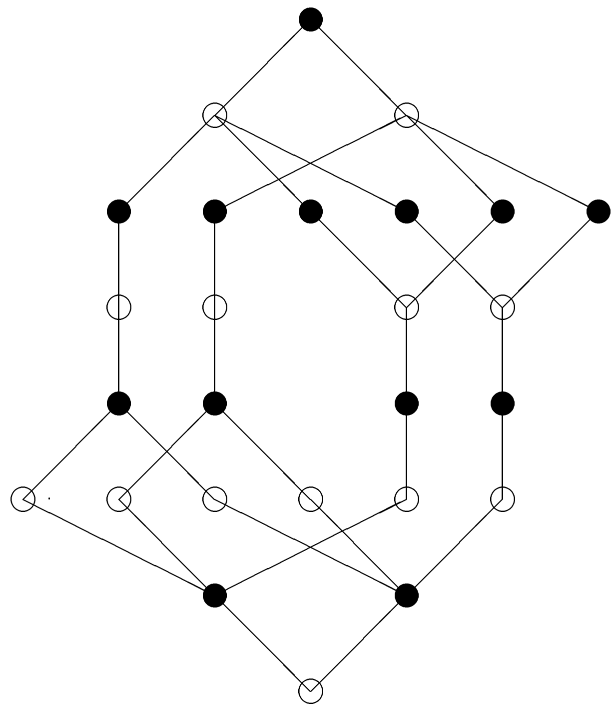

Figure 1: $P\left(6,\{[1,2],[2,6]\}+\{[2,5],[5,6]\}\right)$

A.2 $P\left(6,\{[1,3],[3,4],[4,6]\}+\{[1,2],[2,3]\}+\{[4,5],[5,6]\}\right)$

Take

$$\begin{aligned} P^{I}(N) &= D^{N}_{[1,3]}D^{N}_{[3,4]}D^{N}_{[4,6]}D^{N+1}_{[4,5]}(C^{7}) \\ P^{II}(N) &= D^{N+1}_{[1,2]}D^{N}_{[1,6]}D^{N}_{[2,4]}(C^{7}), \quad \text{and} \\ P^{III}(N) &= D^{N}_{[1,5]}D^{N}_{[3,5]}D^{N}_{[5,6]}(C^{7}). \end{aligned}$$

Identify the elements of $P^{I}(N)$ with the elements of $P^{II}(N)$ at ranks $1,4,5$, and 6. Identify the elements of $P^{I}(N)$ with the elements of $P^{III}(N)$ at ranks $1,2,3$, and 6. Figure 2 represents the resulting poset for $N=2$.

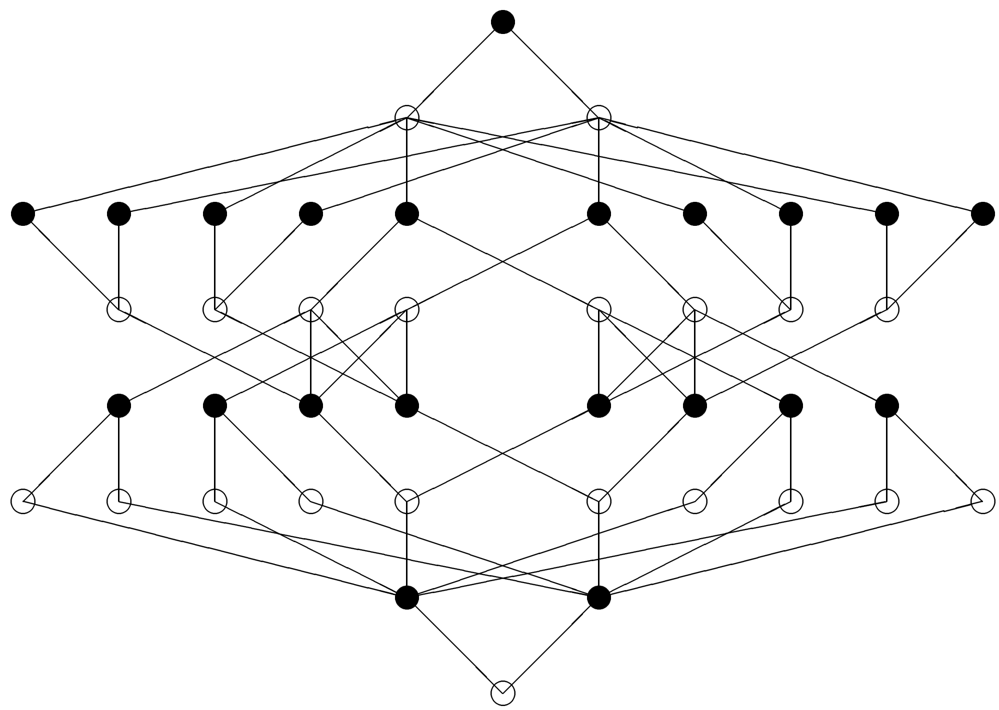

Figure 2: $P\left(6, \{[1,3],[3,4],[4,6]\} + \{[1,2],[2,3]\} + \{[4,5],[5,6]\}\right)$

A.3 $P\left(6, \{[1,2],[3,4],[4,5]\} + \{[3,5],[5,6]\} + \{[1,2],[2,5]\}\right)$

Take

$$
\begin{aligned}
P^{I}(N) &= D_{[1,2]}^{N+1} D_{[3,4]}^{N+1} D_{[3,6]}^{N} D_{[4,5]}^{N+1}(C^7) && \text{(Figure 3)} \\
P^{II}(N) &= D_{[1,5]}^{N+1} D_{[3,5]}^{N^2} D_{[5,6]}^{N}(C^7) && \text{(Figure 4), and} \\
P^{III}(N) &= D_{[1,2]}^{N+2} D_{[2,5]}^{N^2-N+2} D_{[1,6]}^{N}(C^7) && \text{(Figure 5).}
\end{aligned}
$$

Identify the elements of $P^I(N)$ with the elements of $P^{II}(N)$ at ranks $1, 2$, and 6. Identify the elements of $P^I(N)$ with the elements of $P^{III}(N)$ at rank 6. Figure 6 represents the resulting poset for $N = 2$.

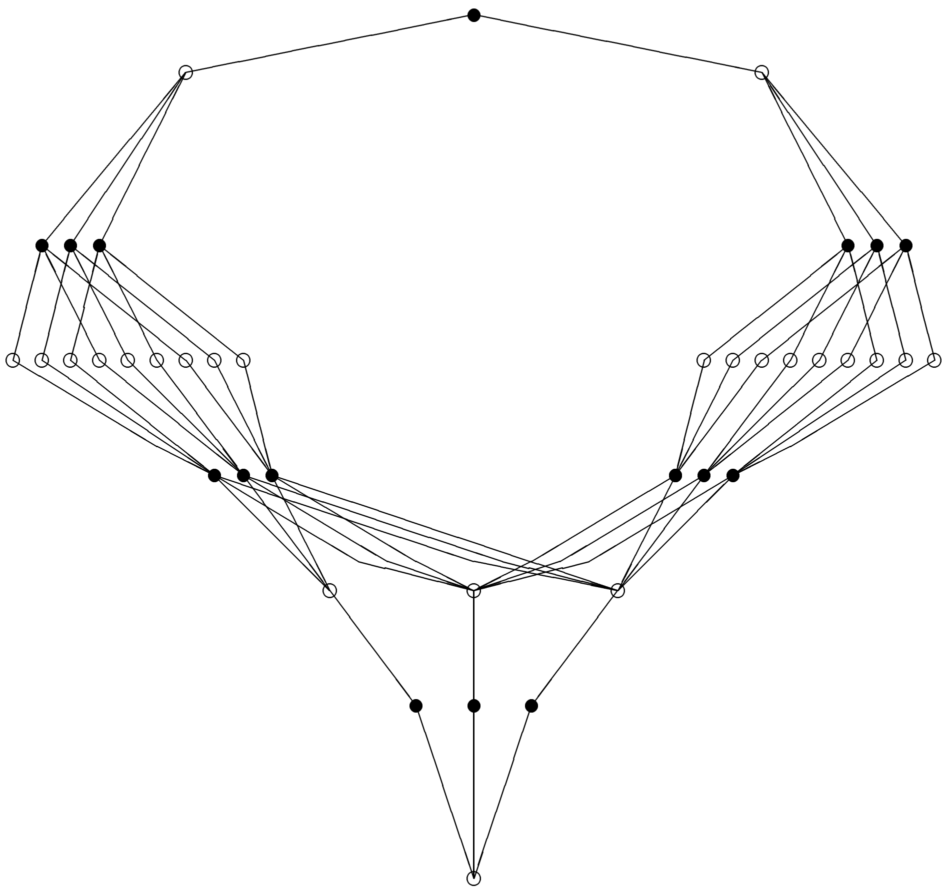

Figure 3: $P^I(2)$

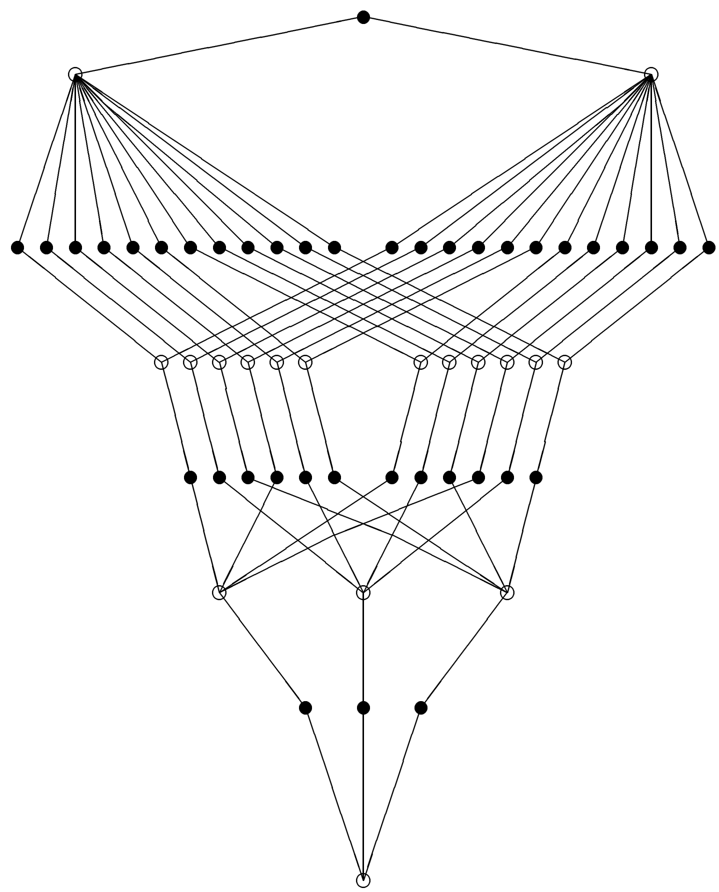

Figure 4: $P^{II}(2)$

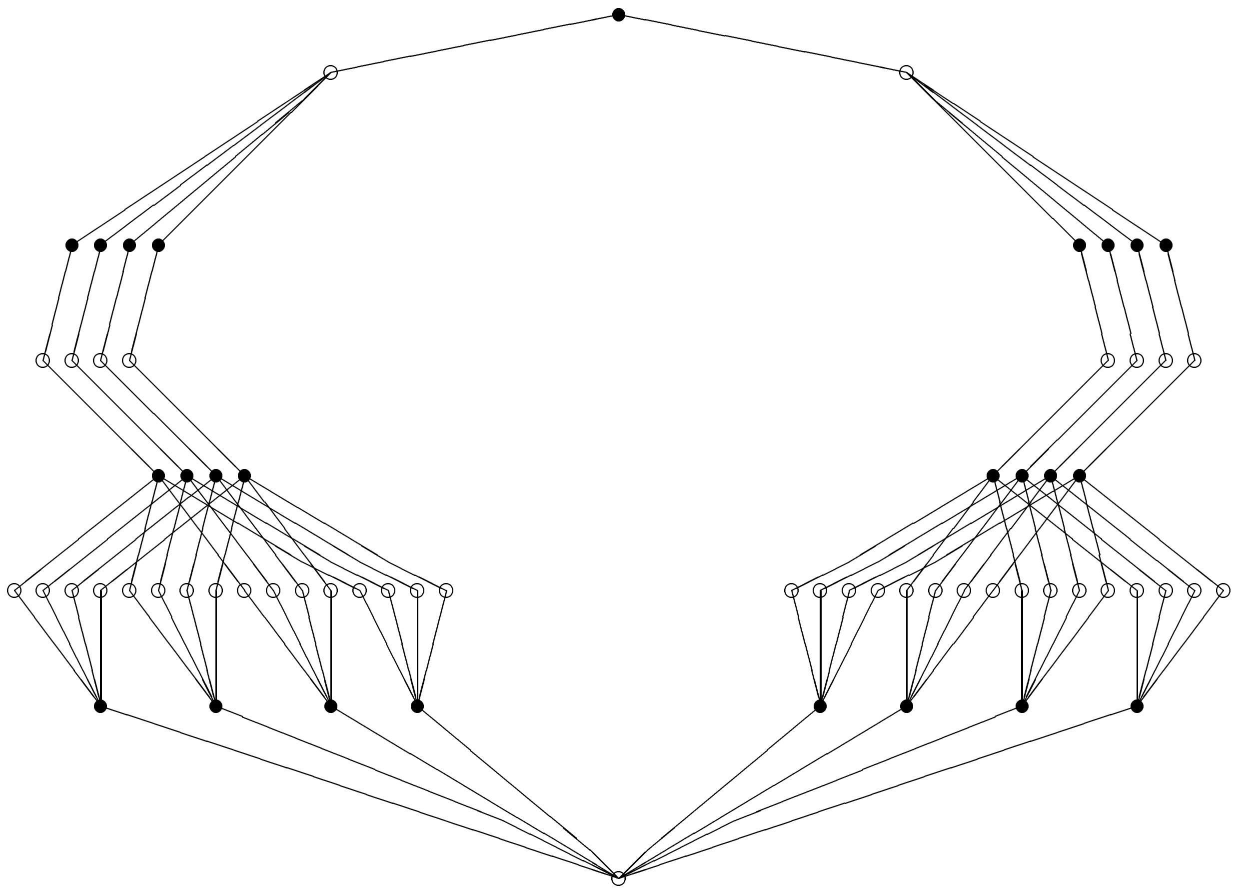

Figure 5: $P^{III}(2)$

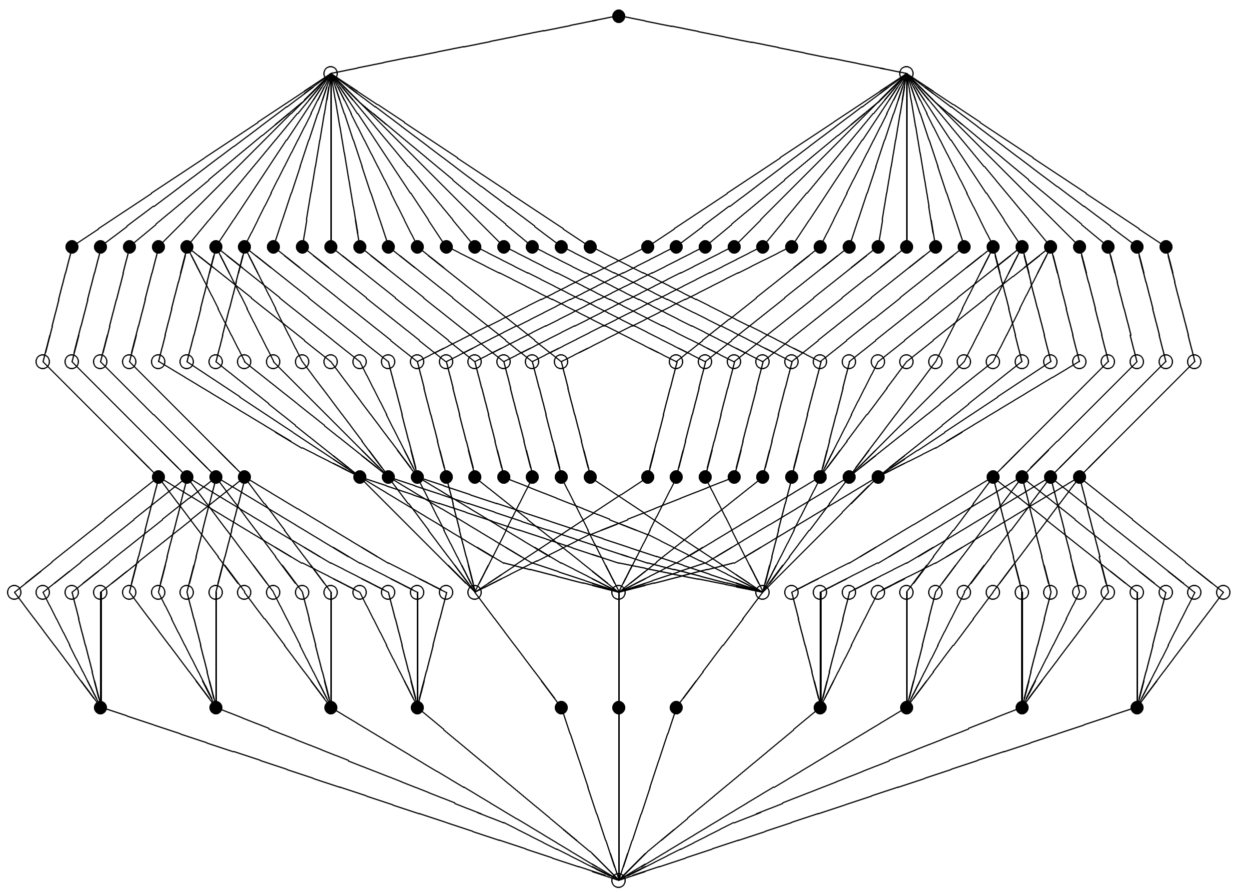

Figure 6: $P\left(6,\{[1,2],[3,4],[4,5]\}+\{[3,5],[5,6]\}+\{[1,2],[2,5]\}\right)$

Appendix B Convolution of inequalities

As in Billera and Liu ([10]) we view the flag f-vector as a vector of *chain operators* $\left(f_S^{n+1} : S \subseteq [1,n]\right)$; here $f_S^{n+1}(P) = f_S(P)$ if P is a graded poset of rank $n+1$ and 0 otherwise. The following multiplication of chain operators f_S^n ($n \geq 1$, $S \subseteq [1, n-1]$) was introduced by Kalai in [15] and studied for Eulerian posets by Billera and Liu in [10]:

$$f_S^m f_T^n = f_{S \cup \{m\} \cup (T+m)}^{m+n}.$$

It is straightforward that given a pair of valid linear inequalities

$$F = \sum_{S \subseteq [1,m-1]} a_S f_S^m \geq 0 \quad \text{and} \quad G = \sum_{T \subseteq [1,n-1]} b_S f_S^n \geq 0$$

that hold for a hereditary class of graded posets, the linear inequality $FG \geq 0$ is also valid for the same class. It was observed by Billera and Liu in [10, Proposition 1.3] that for the class of all graded posets the converse holds as well: if $FG \geq 0$ is a valid inequality, then either both $F \geq 0$ and $G \geq 0$ are valid inequalities, or both $-F \geq 0$ and $-G \geq 0$ are valid inequalities. It is easy to verify that the same equivalence is valid also for the class of (half-)Eulerian posets.

Proposition B.1 *Consider* $F = \sum_{S \subseteq [1,m-1]} a_S f^m$ *and* $G = \sum_{T \subseteq [1,n-1]} b_S f_S^n$. *For these,* $FG \geq 0$ *holds for all half-Eulerian posets if and only if either both* $F \geq 0$ *and* $G \geq 0$ *or both* $-F \geq 0$ *and* $-G \geq 0$ *hold for all half-Eulerian posets. The analogous statement is true for Eulerian posets.*

Only the "only if" implication is not completely trivial. In the half-Eulerian case, all we need to observe is that for a pair (P, Q) of half-Eulerian posets the poset $P \circ Q$ obtained by putting all elements of Q above all elements of P, and identifying the top element of P with the bottom element of Q, is half-Eulerian. Moreover, if for posets P_1, P_2, and Q and forms F and G, $F(P_1) > 0$, $F(P_2) < 0$, and $G(Q) > 0$, then $FG(P_1 \circ Q) = F(P_1)G(Q) > 0$ and $FG(P_2 \circ Q) = F(P_2)G(Q) < 0$. The same argument works for Eulerian posets using $D^2_{\{\rho(P)\}}(P \circ Q)$ instead of $P \circ Q$.

B.1 Unique factorization

According to [10, Theorem 2.1] the associative algebra generated by all chain operators (whose domain is taken to be the class of all graded posets) is the free polynomial ring in variables $\{f_\emptyset^i : i \geq 1\}$. If we take the degree of the variable $f^i_\emptyset$ to be i, then linear combinations of the form $F = \sum_{S\subseteq[1,m-1]} a_S f^m_S$ become homogeneous polynomials. Hence, as noted by Billera and Hetyei in [8], one can use a result of Cohn in [13, Theorem 3] that the semigroup of homogeneous polynomials of a free graded associative algebra has unique factorization. The validity of an inequality may thus be checked factor-by-factor.

For Eulerian and half-Eulerian posets, it is advisable to convert our expressions into the flag-ℓ or flag-L forms, respectively. Straightforward substitution into the definition shows

$$\ell^m_S \ell^n_T = \ell^{m+n}_{S\cup(T+m)} \quad \text{and} \quad L^m_S L^n_T = 2L^{m+n}_{S\cup(T+m)}$$

This means that when we write $[u_S] = L^n_S$ as the coefficient of the ce-word u_S, the convolution of the forms $\sum_{S\subseteq[1,m-1]} a_S[u_S]$ and $\sum_{T\subseteq[1,n-1]} b_T[u_T]$ is the form $2\sum_{S\subseteq[1,m-1]}\sum_{T\subseteq[1,n-1]} a_S b_T[u_S c u_T]$.

Consider the free associative algebra $\mathbf{R}\langle c,e\rangle$ generated by the letters c and e. Given a homogeneous form $F = \sum_{S\subseteq[1,n]} a_S L^{n+1}_S$, set

$$\phi(F) = \frac{1}{2}\sum_{S\subseteq[1,n]}^{n+1} a_S u_S c.$$

Evidently the linear map ϕ is a ring isomorphism between the ring of chain operators (with the convolution operation) and the left ideal $\mathbf{R}\langle c,e\rangle c$ of $\mathbf{R}\langle c,e\rangle$ (with concatenation of letters as multiplication). In terms of this isomorphism we may rephrase [10, Proposition 3.2] as follows:

Proposition B.2 *Let $I_{\mathcal{E}}$ be the two-sided ideal of all forms $\sum_{S\subseteq[1,n]}^{n+1} a_S L^{n+1}_S$ vanishing on all Eulerian posets. Then $\phi(I_{\mathcal{E}})$ is the ideal of $\mathbf{R}\langle c,e\rangle c$ generated by $\{[e^{2k+1}c] \,:\, k\geq 0\}$.*

This statement is a direct consequence of Corollary 1.3. The quotient of $\mathbf{R}\langle c,e\rangle c$ by the ideal $\phi(I_{\mathcal{E}})$ is the left ideal $\mathbf{R}\langle c,ee\rangle c$ of the free noncommutative algebra $\mathbf{R}\langle c,ee\rangle$. By Cohn's result ([13, Theorem 3]) the ring $\mathbf{R}\langle c,ee\rangle$ has unique homogeneous factorization. Given an arbitrary homogeneous expression $E\in\mathbf{R}\langle c,ee\rangle c$, the homogeneous factors c are uniquely identifiable in its unique homogeneous factorization. Hence E may be uniquely written as a product of homogeneous polynomials from $\mathbf{R}\langle c,ee\rangle c$ that are irreducible in $\mathbf{R}\langle c,ee\rangle c$. The analogous observations may be also made in the half-Eulerian setting, and we have the following unique factorization.

Proposition B.3 *Every homogeneous linear form $\sum_{S\subseteq[1,n]} a_S \ell^{n+1}_S$ or $\sum_{S\subseteq[1,n]} a_S L^{n+1}_S$, where S ranges over only even sets, can be uniquely written as a product of irreducible expressions of the same kind.*

B.2 Convolution of facet inequalities

Billera and Hetyei also showed in [8] that for the class of all graded posets the product of two facet inequalities is almost always a facet inequality, every exception being a consequence of the equalities

$$f_\emptyset^m f_\emptyset^n = f_m^{m+n} = \left(f_m^{m+n} - f_\emptyset^{m+n}\right) + f_\emptyset^{m+n}.$$

In terms of convolutions, Proposition 3.1 states that the product of valid inequalities of the form $f_\emptyset^n \geq 0$ and $f_i^n - 2f_\emptyset^n \geq 0$ is a valid inequality for all Eulerian posets. Theorem 4.6 describes a subclass of these products that yield facet inequalities. Using ideas extracted from the proof, one can show the following, somewhat strengthened statements.

Proposition B.4 *If $F \geq 0$ defines a facet of $\mathcal{C}_\mathcal{E}^{n+1}$, then $F(f_1^{k+1} - 2f_\emptyset^{k+1}) \geq 0$ defines a facet of $\mathcal{C}_\mathcal{E}^{n+k+2}$.*

Proposition B.5 *If $F \geq 0$ defines a facet of $\mathcal{C}_\mathcal{E}^{n+1}$, and F can be written as*

$$F = \sum_{S \subseteq [1,n]} a_S L_S^{n+1}$$

where S ranges over only even sets that contain n, then $F f_\emptyset^{k+1} \geq 0$ and $F f_\emptyset^1 f_\emptyset^1 \geq 0$ define facets of $\mathcal{C}_\mathcal{E}^{n+k+2}$ and $\mathcal{C}_\mathcal{E}^{n+3}$, respectively.

It seems to be difficult, however, even in the case of these simple factors, to predict which products yield facet inequalities. For example $(f_1^5 - 2f_\emptyset^5)f_\emptyset^1 = (f_1^6 - 2f_\emptyset^6) + \frac{1}{2}(f_1^3 - 2f_\emptyset^3)(f_1^3 - 2f_\emptyset^3) \geq 0$ does not define a facet of $\mathcal{C}_\mathcal{E}^6$, while it can be shown that $(f_1^5 - 2f_\emptyset^5)f_\emptyset^3 \geq 0$ defines a facet of $\mathcal{C}_\mathcal{E}^8$.

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