Facets of the cone up to rank 7

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April 28, 2000

This note is available on the Worldwide Web at the address

`http://www.math.ku.edu/~bayer/euler/index.html`

and it provides supplemental information to our paper *"Flag vectors of Eulerian partially ordered sets"*, to appear in the European Journal of Combinatorics.

1 Description of notation

Convolution of chain operators: $f_S^m f_T^n \stackrel{\text{def}}{=} f_{S\cup\{m\}\cup(T+m)}^{m+n}$.
See Appendix B of our paper.

Sparse f basis: A set $S \subseteq \{1, 2, \ldots, n\}$ is *sparse* if it does not contain two consecutive integers, and it does not contain n. For the vector space of chain operators $\langle f_S^{n+1} : S \subseteq \{1, 2, \ldots, n\}\rangle$ acting on Eulerian posets of rank $n + 1$, the set $\{f_S^{n+1} : S \subseteq \{1, 2, \ldots, n\}, S \text{ sparse}\}$ forms a basis. This was shown in:

M. M. Bayer and L. J. Billera, Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.

L-vector: $L_S^{n+1} \stackrel{\text{def}}{=} (-1)^{n-|S|} \sum_{T\supseteq[1,n]\setminus S} \left(-\frac{1}{2}\right)^{|T|} f_T^{n+1}$. Equivalently $f_S^{n+1} = 2^{|S|} \sum_{T\subseteq[1,n]\setminus S} L_T^{n+1}$.
See Definition 2 of our paper.

*This research was supported by University of Kansas General Research allocation #3552.
†On leave from the Alfréd Rényi Institute. Partially supported by Hungarian National Foundation for Scientific Research grant no. F 032325.

The cones: $\mathcal{C}_{\mathcal{E}}^{n+1}$ is the smallest closed convex cone containing the flag vectors of all Eulerian posets.

$\mathcal{C}_{\mathcal{D}}^{n+1}$ is the smallest closed convex cone containing the flag vectors of the horizontal doubles of all half-Eulerian posets. (See Section 2 of our paper.)

2 From PORTA output to Eulerian cone

Let $\mathcal{V}$ be a set of vectors contained in the cone of half-Eulerian flag f-vectors, and let cone($\mathcal{V}$) be the smallest convex cone containing $\mathcal{V}$. Consider the affine transformation ϕ that multiplies each f_S-entry of a vector by $2^{|S|}$, and divides the coefficient of f_S in an inequality by $2^{|S|}$. Let $\mathcal{K}^{n+1}$ be the image of cone($\mathcal{V}$) under ϕ. Then $\mathcal{K}^{n+1}$ is contained in the closed cone $\mathcal{C}_{\mathcal{D}}^{n+1}$ of flag f-vectors of horizontal doubles of half-Eulerian posets, which is contained in the Eulerian cone $\mathcal{C}_{\mathcal{E}}^{n+1}$.

For each rank $n+1 \leq 7$ we created a finite set $\mathcal{V}$ in the cone of half-Eulerian flag f-vectors, and used PORTA to generate cone($\mathcal{V}$). In this note we list the facet inequalities for the corresponding cone $\mathcal{K}^{n+1}$ ($n \leq 6$). We check that these inequalities are valid for all Eulerian posets (not just for doubled half-Eulerian posets), and this tells us that $\mathcal{C}_{\mathcal{E}}^{n+1} \subseteq \mathcal{K}^{n+1}$ for rank at most 7. Thus $\mathcal{K}^{n+1} = \mathcal{C}_{\mathcal{D}}^{n+1} = \mathcal{C}_{\mathcal{E}}^{n+1}$, so we have in fact determined the closed cone of Eulerian flag f-vectors for rank $n+1 \leq 7$.

3 Inequalities in the sparse f basis

The inequalities listed are equivalent up to a constant factor to the inequalities encoded in the corresponding PORTA output files. The second number in brackets corresponds to the number of the equivalent inequality in the L-basis. Boldfaced inequalities cannot be written as convolutions of lower-rank inequalities.

Rank 1:

$$(F1) \quad \mathbf{f}_{\emptyset}^{1} \geq 0 \quad (L1)$$

Rank 2:

$$(F1) \quad f_{\emptyset}^{2} \geq 0 \quad (L1)$$

(F1) is equivalent to $\frac{1}{2} f_1^2 = \frac{1}{2} f_{\emptyset}^1 f_{\emptyset}^1 \geq 0$.

Rank 3:

$$\begin{array}{lrl} (F1) & \mathbf{f}_{\emptyset}^{3} \geq 0 & (L2) \\ (F2) & -2f_{\emptyset}^{3} + f_1^3 \geq 0 & (L1) \end{array}$$

Rank 4:

$$\begin{array}{lrl} (F1) & \mathbf{f}^4_\emptyset \geq 0 & (L3) \\ (F2) & -f^4_1 + f^4_2 \geq 0 & (L2) \\ (F3) & -2f^4_\emptyset + f^4_1 \geq 0 & (L1) \end{array}$$

Equivalent forms of reducible or non-evident inequalities:

$(F2)$: $\frac{1}{2}(f^4_{12} - 2f^4_1) = \frac{1}{2}f^1_\emptyset(f^3_1 - 2f^3_\emptyset) \geq 0.$

$(F3)$: $f^4_2 - f^4_3 = \frac{1}{2}(f^4_{13} - 2f^4_3) = \frac{1}{2}(f^3_1 - 2f^3_\emptyset)f^1_\emptyset \geq 0.$

Rank 5:

$$\begin{array}{lrl} (F1) & \mathbf{f}^5_\emptyset \geq 0 & (L6) \\ (F2) & -2f^5_3 + f^5_{13} \geq 0 & (L3) \\ (F3) & -2f^5_2 + f^5_{13} \geq 0 & (L1) \\ (F4) & -f^5_1 + f^5_2 \geq 0 & (L2) \\ (F5) & -2f^5_\emptyset + f^5_1 \geq 0 & (L5) \\ (F6) & -2f^5_\emptyset + f^5_1 - f^5_2 + f^5_3 \geq 0 & (L4) \end{array}$$

Equivalent forms of reducible or non-evident inequalities:

$(F2)$: $(f^3_1 - 2f^3_\emptyset)f^2_\emptyset = \frac{1}{2}(f^3_1 - 2f^3_\emptyset)f^1_\emptyset f^1_\emptyset \geq 0.$

$(F3)$: $\frac{1}{2}(f^5_{123} - 4f^5_2) = \frac{1}{2}(f^5_{123} - 2f^5_{12}) = \frac{1}{2}f^1_\emptyset f^1_\emptyset(f^3_1 - 2f^3_\emptyset) \geq 0.$

$(F4)$: $\frac{1}{2}(f^5_{12} - 2f^5_1) = \frac{1}{2}f^1_\emptyset(f^4_1 - 2f^4_\emptyset) = \frac{1}{4}f^1_\emptyset(f^3_1 - 2f^3_\emptyset)f^1_\emptyset \geq 0.$

$(F6)$: $\mathbf{f}^5_4 - 2f^5_\emptyset \geq 0.$

Rank 6:

$$\begin{array}{lrl} (F1) & \mathbf{f}^6_\emptyset \geq 0 & (L10) \\ (F2) & -f^6_{14} + f^6_{24} \geq 0 & (L2) \\ (F3) & -2f^6_3 + f^6_{13} \geq 0 & (L6) \\ (F4) & -2f^6_2 + f^6_{13} \geq 0 & (L1) \\ (F5) & -f^6_1 + f^6_2 \geq 0 & (L5) \\ (F6) & -2f^6_\emptyset + f^6_1 \geq 0 & (L8) \\ (F7) & 2f^6_3 - f^6_{13} - 2f^6_4 + f^6_{14} \geq 0 & (L9) \\ (F8) & -2f^6_3 + 2f^6_4 - f^6_{14} + f^6_{24} \geq 0 & (L3) \\ (F9) & -f^6_1 + f^6_2 - f^6_3 + f^6_4 \geq 0 & (L7) \\ (F10) & -2f^6_\emptyset + f^6_1 - f^6_2 + f^6_3 \geq 0 & (L4) \end{array}$$

Equivalent forms of reducible or non-evident inequalities:

$(F2)$: $(f^4_1 - f^4_2)f^2_\emptyset = \frac{1}{4}f^1_\emptyset(f^3_1 - 2f^3_\emptyset)f^1_\emptyset f^1_\emptyset \geq 0.$

$(F3)$: $(f^3_1 - f^3_\emptyset)f^3_\emptyset \geq 0.$

$(F4)$: $\frac{1}{2}(-2f^6_{12} + f^6_{123}) = \frac{1}{2}f^1_\emptyset f^1_\emptyset(2f^4_\emptyset - f^4_1) = \frac{1}{4}f^1_\emptyset f^1_\emptyset(f^3_1 - 2f^3_\emptyset)f^1_\emptyset \geq 0.$

$(F5)$: $\frac{1}{2}(f^6_{12} - 2f^6_1) = \frac{1}{2}f^1_\emptyset(f^5_1 - 2f^5_\emptyset).$

$(F7): 2f_3^6 - f_{23}^6 + f_{24}^6 - f_{34}^6 = \frac{1}{2}(4f_3^6 - 2f_{23}^6 + f_{234}^6 - 2f_{34}^6) = \frac{1}{2}(f_2^3 - 2f_\emptyset^3)(f_1^3 - 2f_\emptyset^3)$
$= \frac{1}{2}(f_1^3 - 2f_\emptyset^3)(f_1^3 - 2f_\emptyset^3) \geq 0.$

$(F8): f_{34}^6 - 2f_3^6 = f_\emptyset^3(f_1^3 - 2f_\emptyset^3) \geq 0.$

$(F9): \mathbf{f}_5^6 - 2f_\emptyset^6 \geq 0.$

$(F10): f_4^6 - f_5^6 = \frac{1}{2}(f_{45}^6 - 2f_5^6) = \frac{1}{2}(f_4^5 - 2f_\emptyset^5)f_\emptyset^1 \geq 0.$

Rank 7:

$(F1)$	$\mathbf{f}_\emptyset^7 \geq 0$	$(L23)$
$(F2)$	$-2f_{25}^7 + f_{135}^7 \geq 0$	$(L1)$
$(F3)$	$-f_{14}^7 + f_{24}^7 \geq 0$	$(L5)$
$(F4)$	$-2f_3^7 + f_{13}^7 \geq 0$	$(L10)$
$(F5)$	$-2f_2^7 + f_{13}^7 \geq 0$	$(L3)$
$(F6)$	$-f_1^7 + f_2^7 \geq 0$	$(L9)$
$(F7)$	$-2f_\emptyset^7 + f_1^7 \geq 0$	$(L13)$
$(F8)$	$-2f_\emptyset^7 + f_3^7 \geq 0$	$(L15)$
$(F9)$	$-2f_\emptyset^7 + f_4^7 \geq 0$	$(L14)$
$(F10)$	$2f_3^7 - f_{13}^7 - 2f_4^7 + f_{14}^7 \geq 0$	$(L16)$
$(F11)$	$4f_4^7 - 2f_{14}^7 - 2f_{35}^7 + f_{135}^7 \geq 0$	$(L17)$
$(F12)$	$f_{14}^7 - f_{24}^7 - f_{15}^7 + f_{25}^7 \geq 0$	$(L18)$
$(F13)$	$-2f_5^7 + f_{15}^7 - f_{25}^7 + f_{35}^7 \geq 0$	$(L4)$
$(F14)$	$-2f_4^7 + f_{15}^7 - f_{25}^7 + f_{35}^7 \geq 0$	$(L7)$
$(F15)$	$-2f_3^7 + 2f_4^7 - f_{14}^7 + f_{24}^7 \geq 0$	$(L2)$
$(F16)$	$-2f_2^7 + 2f_3^7 - 2f_5^7 + f_{25}^7 \geq 0$	$(L22)$
$(F17)$	$-2f_2^7 + f_{13}^7 - f_{14}^7 + f_{15}^7 \geq 0$	$(L6)$
$(F18)$	$-2f_2^7 + 2f_4^7 - 2f_5^7 + f_{25}^7 \geq 0$	$(L21)$
$(F19)$	$-f_1^7 + f_2^7 - f_3^7 + f_4^7 \geq 0$	$(L8)$
$(F20)$	$-2f_3^7 + f_{13}^7 + 2f_4^7 - f_{14}^7 - 2f_5^7 + f_{15}^7 \geq 0$	$(L11)$
$(F21)$	$-2f_1^7 + f_{13}^7 + 2f_4^7 - f_{14}^7 - 2f_5^7 + f_{15}^7 \geq 0$	$(L20)$
$(F22)$	$-2f_\emptyset^7 + f_1^7 - f_2^7 + f_3^7 - f_4^7 + f_5^7 \geq 0$	$(L12)$
$(F23)$	$-2f_1^7 - 2f_3^7 + f_{13}^7 + 4f_4^7 - f_{14}^7 - 2f_5^7 + f_{15}^7 \leq 0$	$(L19)$

Equivalent forms of reducible or non-evident inequalities:

$(F2)$: $(-2f_2^5 + f_{13}^5)f_\emptyset^2 = \frac{1}{4}f_\emptyset^1 f_\emptyset^1(f_1^3 - 2f_\emptyset^3)f_\emptyset^1 f_\emptyset^1 \geq 0.$

$(F3)$: $(f_2^4 - f_1^4)f_\emptyset^3 = \frac{1}{2}f_\emptyset^1(f_1^3 - 2f_\emptyset^3)f_\emptyset^3 \geq 0.$

$(F4)$: $(f_1^3 - 2f_\emptyset^3)f_\emptyset^4 \geq 0.$

$(F5)$: $\frac{1}{2}(-2f_{12}^7 + f_{123}^7) = \frac{1}{2}f_\emptyset^1 f_\emptyset^1(f_1^5 - 2f_\emptyset^5) \geq 0.$

$(F6)$ $\frac{1}{2}(-2f_1^7 + f_{12}^7) = \frac{1}{2}f_\emptyset^1(f_1^6 - 2f_\emptyset^6) \geq 0.$

$(F10)$: $2f_3^7 - f_{23}^7 + f_{24}^7 - f_{34}^7 = \frac{1}{2}(4f_3^7 - 2f_{23}^7 + f_{234}^7 - 2f_{34}^7) = \frac{1}{2}(f_1^3 - 2f_\emptyset^3)(f_1^4 - 2f_\emptyset^4)$
$= \frac{1}{4}(f_1^3 - 2f_\emptyset^3)(f_1^3 - 2f_\emptyset^3)f_\emptyset^1 \geq 0.$

$(F11)$: $-2f_{24}^7+2f_{34}^7-2f_{35}^7+f_{135}^7 = \frac{1}{2}(-2f_{134}^7+4f_{34}^7-2f_{345}^7+f_{1345}^7) = \frac{1}{2}(f_1^3-2f_\emptyset^3)f_\emptyset^1(f_1^3-2f_\emptyset^3) \geq 0.$

$(F12)$: $2f_4^7 - f_{34}^7 + f_{35}^7 - f_{45}^7 = \frac{1}{2}(4f_4^7 - 2f_{34}^7 + f_{345}^7 - 2f_{45}^7) = \frac{1}{2}(f_3^4 - 2f_\emptyset^4)(f_1^3 - 2f_\emptyset^3)$
$= \frac{1}{2}(f_2^4 - f_1^4)(f_1^3 - 2f_\emptyset^3) = \frac{1}{4}f_\emptyset^1(f_2^3 - 2f_\emptyset^3)(f_1^3 - 2f_\emptyset^3) = \frac{1}{4}f_\emptyset^1(f_1^3 - 2f_\emptyset^3)(f_1^3 - 2f_\emptyset^3) \geq 0.$

$(F13)$: $f_{45}^7 - 2f_5^7 = \frac{1}{2}(f_4^5 - 2f_\emptyset^5)f_\emptyset^1 f_\emptyset^1 \geq 0.$

$(F14)$: $f_{45}^7 - 2f_4^7 = f_\emptyset^4(f_1^3 - 2f_\emptyset^3) \geq 0.$

$(F15)$: $f_{34}^7 - 2f_3^7 = f_\emptyset^3(f_1^4 - 2f_\emptyset^4) = \frac{1}{2}f_\emptyset^3(f_1^3 - 2f_\emptyset^3)f_\emptyset^1 \geq 0.$

$(F17)$: $f_{16}^7 - 2f_1^7 = f_\emptyset^1(f_5^6 - 2f_\emptyset^6) \geq 0.$

$(F19)$: $f_5^7 - f_6^7 = \frac{1}{2}(f_{56}^7 - 2f_6^7) = \frac{1}{2}(f_5^6 - 2f_\emptyset^6)f_\emptyset^1 \geq 0.$

$(F20)$: $f_{16}^7 - 2f_6^7 = (f_1^6 - 2f_\emptyset^6)f_\emptyset^1 \geq 0.$

$(F21)$: $\mathbf{f}_{16}^7 - 2f_1^7 - 2f_6^7 + 2f_3^7 \geq 0.$

$(F22)$: $\mathbf{f}_6^7 - 2f_\emptyset^7 \geq 0.$

$(F23)$: $\mathbf{f}_{16}^7 - 2f_1^7 - 2f_6^7 + 2f_4^7 \geq 0.$

4 Inequalities in the L-basis

Note: Since $f_S^{n+1} = 2^{|S|} \sum_{T\subseteq[1,n]\setminus S} L_T^{n+1}$, and here we need to consider only the nonzero L_T^{n+1}'s, it is easy to convert an f-formula into an L-formula. The other way may be more difficult. The labels (Fi) refer to the numbering in the sparse f basis which is given in section 3. Boldfaced inequalities cannot be written as convolutions of lower-rank inequalities.

Rank 1:

$$(L1) \quad -L_\emptyset^1 \leq 0 \quad (F1)$$

Rank 2:

$$(L1) \quad -L_\emptyset^2 \leq 0 \quad (F1)$$

Rank 3:

$$\begin{array}{lrl} (L1) & L^3_{12} \leq 0 & (F2) \\ (L2) & -L^3_{\emptyset} - L^3_{12} \leq 0 & (F1) \end{array}$$

Rank 4:

$$\begin{array}{lrl} (L1) & L^4_{12} = L^3_{12} L^1_{\emptyset} \leq 0 & (F3) \\ (L2) & L^4_{23} = L^1_{\emptyset} L^3_{12} \leq 0 & (F2) \\ (L3) & -L^4_{\emptyset} - L^4_{12} - L^4_{23} \leq 0 & (F1) \end{array}$$

Rank 5:

$$\begin{array}{lrl} (L1) & L^5_{34} \leq 0 & (F3) \\ (L2) & L^5_{23} \leq 0 & (F4) \\ (L3) & L^5_{12} \leq 0 & (F2) \\ (L4) & L^5_{34} + L^5_{1234} \leq 0 & (F6) \\ (L5) & L^5_{12} + L^5_{1234} \leq 0 & (F5) \\ (L6) & \boldsymbol{-L^5_{\emptyset} - L^5_{12} - L^5_{23} - L^5_{34} - L^5_{1234} \leq 0} & (F1) \end{array}$$

Rank 6:

$$\begin{array}{lrl} (L1) & L^6_{34} \leq 0 & (F4) \\ (L2) & L^6_{23} \leq 0 & (F2) \\ (L3) & L^6_{45} + L^6_{1245} \leq 0 & (F8) \\ (L4) & L^6_{34} + L^6_{1234} \leq 0 & (F10) \\ (L5) & L^6_{23} + L^6_{2345} \leq 0 & (F5) \\ (L6) & L^6_{12} + L^6_{1245} \leq 0 & (F3) \\ (\mathbf{L7}) & \boldsymbol{L^6_{45} + L^6_{1245} + L^6_{2345} \leq 0} & (F9) \\ (\mathbf{L8}) & \boldsymbol{L^6_{12} + L^6_{1234} + L^6_{1245} \leq 0} & (F6) \\ (L9) & -L^6_{1245} \leq 0 & (F7) \\ (\mathbf{L10}) & -L^{\mathbf{6}}_{\emptyset} - L^{\mathbf{6}}_{\mathbf{12}} - L^{\mathbf{6}}_{\mathbf{23}} - L^{\mathbf{6}}_{\mathbf{34}} - L^{\mathbf{6}}_{\mathbf{1234}} - L^{\mathbf{6}}_{\mathbf{45}} - L^{\mathbf{6}}_{\mathbf{1245}} - L^{\mathbf{6}}_{\mathbf{2345}} \leq 0 & (F10) \end{array}$$

Rank 7:

$$
\begin{array}{lrr}
(L1) & L^7_{34} \le 0 & (F2) \\
(L2) & L^7_{45} + L^7_{1245} \le 0 & (F15) \\
(L3) & L^7_{34} + L^7_{3456} \le 0 & (F5) \\
(L4) & L^7_{34} + L^7_{1234} \le 0 & (F13) \\
(L5) & L^7_{23} + L^7_{2356} \le 0 & (F3) \\
(L6) & L^7_{56} + L^7_{2356} + L^7_{3456} \le 0 & (F17) \\
(L7) & L^7_{56} + L^7_{1256} + L^7_{2356} \le 0 & (F14) \\
(L8) & L^7_{45} + L^7_{1245} + L^7_{2345} \le 0 & (F19) \\
(L9) & L^7_{23} + L^7_{2345} + L^7_{2356} \le 0 & (F6) \\
(L10) & L^7_{12} + L^7_{1245} + L^7_{1256} \le 0 & (F4) \\
(L11) & L^7_{12} + L^7_{1234} + L^7_{1245} \le 0 & (F20) \\
(L12) & L^7_{56} + L^7_{1256} + L^7_{2356} + L^7_{3456} + L^7_{123456} \le 0 & (F22) \\
(L13) & L^7_{12} + L^7_{1234} + L^7_{1245} + L^7_{1256} + L^7_{123456} \le 0 & (F7) \\
(L14) & L^7_{34} + L^7_{1234} + L^7_{45} + L^7_{1245} + L^7_{2345} + L^7_{3456} + L^7_{123456} \le 0 & (F9) \\
(L15) & L^7_{23} + L^7_{34} + L^7_{1234} + L^7_{2345} + L^7_{2356} + L^7_{3456} + L^7_{123456} \le 0 & (F8) \\
(L16) & -L^7_{1245} \le 0 & (F10) \\
(L17) & -L^7_{1256} \le 0 & (F11) \\
(L18) & -L^7_{2356} \le 0 & (F12) \\
(L19) & L^7_{34} + L^7_{1234} + L^7_{45} + L^7_{1245} + L^7_{2345} - L^7_{1256} + L^7_{3456} \le 0 & (F23) \\
(L20) & L^7_{23} + L^7_{34} + L^7_{1234} + L^7_{2345} - L^7_{1256} + L^7_{2356} + L^7_{3456} \le 0 & (F21) \\
(L21) & L^7_{34} + L^7_{1234} + L^7_{45} - L^7_{1256} - L^7_{2356} + L^7_{3456} \le 0 & (F18) \\
(L22) & L^7_{23} + L^7_{34} + L^7_{1234} - L^7_{1245} - L^7_{1256} + L^7_{3456} \le 0 & (F16) \\
(L23) & -L^7_{\emptyset} - L^7_{12} - L^7_{23} - L^7_{34} - L^7_{1234} - L^7_{45} & (F1) \\
 & -L^7_{1245} - L^7_{2345} - L^7_{56} - L^7_{1256} - L^7_{2356} - L^7_{3456} - L^7_{123456} \le 0 &
\end{array}
$$