Flag Vectors of Polytopes

An Overview

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Definitions

CONVEX POLYTOPE:

$$P = \mathrm{conv}\{x_1, x_2, \ldots, x_n\} \subset \mathbf{R}^d$$

proper FACE:

intersection of supporting hyperplane with P

FACE LATTICE:

$\emptyset$, P, and proper faces, ordered by inclusion

FACE VECTOR:

$$(f_0(P), f_1(P), \ldots, f_{d-1}(P))$$

$f_i(P) = \#$ of i-dimensional faces of P

The main problem

BIG PROBLEM:

Characterize the face vectors of d-dimensional convex polytopes.

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THEOREM (STEINITZ)

$(f_0, f_1, f_2) \in \mathbf{N}^3$ is the face vector of a 3-dimensional convex polytope if and only if

1. $f_0 - f_1 + f_2 = 2$ and
2. $2f_1 \geq 3f_0$ and $2f_1 \geq 3f_2$.

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THEOREM (STEINITZ) ⋆ HAPPY 100TH ANNIVERSARY ⋆

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The g-theorem

THEOREM (Conjectured by McMullen; proved by Stanley, and Billera and Lee 1980)

Characterization of all face vectors of simplicial polytopes

- linear equations (Dehn-Sommerville)
- linear inequalities
- nonlinear inequalities

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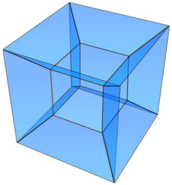
NONSIMPLICIAL, dim $\geq$ 4?

still open
need to look further than the face vector ...



Flag vectors

Let $S = \{s_1, s_2, \ldots, s_k\}_< \subseteq \{0, 1, \ldots, d-1\}$.

Definition

An S-flag of P is a chain

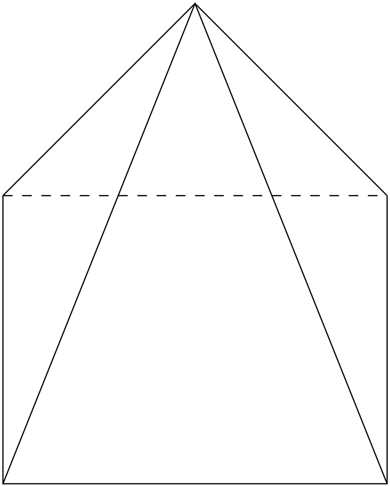
$$\emptyset \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset P$$

with $\dim F_i = s_i$.

$f_S(P) = \#$ of S-flags of P

$(f_S(P))_{S \subseteq \{0,1,\ldots,d-1\}}$ is the flag vector of P.

Example



$f_\emptyset = 1$
$f_0 = 5$
$f_1 = 8$
$f_2 = 5$
$f_{01} = 16$
$f_{02} = 16$
$f_{12} = 16$
$f_{012} = 32$

Why Study Flag Vectors of Polytopes?

- Stanley (1970s) studied (f_S) for balanced simplicial complexes/order complexes of graded posets.
- For 3-dimensional polytopes and simplicial polytopes, for which the face vectors are characterized, the flag vector depends (linearly) on the face vector.
- For general polytopes, the flag vector reflects greater combinatorial complexity than the face vector.
- Inequalities on flag vectors project to inequalities on face vectors.
- Flag vectors relate to parameters from algebraic geometry.

Generalized Dehn-Sommerville Equations

THEOREM (B-Billera 1983)

The dimension of the linear span of flag vectors of d-polytopes is the dth Fibonacci number.

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Independent flag numbers

- dimension 3: $f_\emptyset$, f_0, f_1
- dimension 4: $f_\emptyset$, f_0, f_1, f_2, f_{02}
- dimension 5: $f_\emptyset$, f_0, f_1, f_2, f_3, f_{02}, f_{03}, f_{13}

Inequalities on Flag Vectors

Kalai rigidity inequality (1987)

$$f_{02} - 3f_2 + f_1 - df_0 + \binom{d+1}{2} \geq 0$$

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4-dimensional polytopes (B 1987)

$$\begin{aligned}
&f_0 \geq 5\\
&f_3 \geq 5\\
&f_{02} - 3f_2 \geq 0\\
&f_{02} - 3f_1 \geq 0\\
&f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0\\
&6f_1 - 6f_0 - f_{02} \geq 0
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Compare: face vectors of 4-polytopes by Barnette, Grünbaum, Reay, 1967–1974.

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Compare: face vectors of 4-polytopes by Barnette, Grünbaum, Reay, 1967–1974.

Not known to be best possible.
No further linear inequalities for $d = 4$ since 1987.

Toric h-vector (Stanley 1987)

$(h_0, h_1, h_2, \ldots, h_d)$

middle perversity intersection homology Betti numbers

$(h_0, h_1, h_2, \ldots, h_d)$ depends linearly on flag vector

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$1 = h_0 \leq h_1 \leq h_2 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$

from algebraic geometry (for rational polytopes)

imply linear inequalities on flag vectors

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Nonlinear inequalities?

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Karu 2004 broke dependence on algebraic geometry, extending results to nonrational polytopes.

cd-index

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vector of length = dth Fibonacci number
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Examples

4-simplex: $cccc + 3dcc + 5cdc + 3ccd + 4dd$
4-cube: $cccc + 14dcc + 16cdc + 6ccd + 20dd$

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Fine conjecture

Each coefficient in the *cd*-index of a polytope is ≥ 0.
Proved by Stanley 1994 (for S-shellable spheres)
Strengthened by Billera and Ehrenborg 2000: The d-simplex minimizes each coefficient among d-polytopes.

Linear inequalities on flag vectors

Basic linear inequalities come from

- toric *h*-vector
- *cd*-index

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- Which are redundant?
- Which give facets of the closed convex cone of flag vectors?

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Some answers by . . .

- Ehrenborg 2005
- Stenson 2004, 2005

Kalai's convolution

Definition

$$S \subseteq \{0, 1, \ldots, d-1\} \qquad T \subseteq \{0, 1, \ldots, e-1\}$$

$$\begin{aligned} f_S * f_T(P) &= \sum_{\dim F = d} f_S(F) f_T(P/F) \\ &= f_{S \cup \{d\} \cup (T + (d+1))}(P) \end{aligned}$$

flag number of $(d+e+1)$-dimensional polytope

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Convolutions produce inequalities

If m_d is a nonnegative linear form in f_S, $S \subseteq \{0, 1, \ldots, d-1\}$, and n_e is a nonnegative linear form in f_T, $T \subseteq \{0, 1, \ldots, e-1\}$, then $m_d * n_e \geq 0$ for $(d+e+1)$-polytopes.

Ehrenborg's lifting

Example

For a *cd*-word w, and a convex polytope P, write $[w]$ for the coefficient of w in the *cd*-index of P.

Kalai's rigidity inequality for 4-polytopes

$$f_{02}(P) - 3f_2(P) + f_1(P) - 4f_0(P) + 10 \geq 0$$

can be written in terms of the *cd*-index, as

$$[dd] - [ccd] - [dcc] + 2[cccc] \geq 0.$$

Ehrenborg lifting then gives:
For every *cd*-words u and v where u does not end in c and $\deg u + \deg v = n$, for every $(n+4)$-dimensional polytope

$$[uddv] - [uccdv] - [udccv] + 2[uccccv] \geq 0.$$

Flag vectors of 4-dimensional polytopes

toric h-vector and cd-index don't give new linear inequalities for 4-polytopes

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Ziegler

focus on fatness/complexity gives better understanding, suggests directions and constructions

Fatness

$$F(P) = \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}$$

$5/2 \leq F(P)$
Is there an upper bound for $F(P)$?
Largest known $F(P) < 9$

New results on 4-polytopes!

Paffenholz and Werner 2006

construction of infinite family of 4-polytopes that are 2-simplicial, 2-simple, and elementary
gives extreme ray of cone of flag vectors

Definitions

A polytope is 2-simplicial if every 2-face is a triangle.
A polytope is 2-simple if every edge is contained in exactly 3 facets.
A polytope P is elementary if $f_{02}(P) - 3f_2(P) + f_1(P) - 4f_0(P) + 10 = 0$

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Other examples of 2-simplicial, 2-simple polytopes (Eppstein, Kuperberg, Paffenholz, Ziegler)

Recall inequalities on 4-dimensional polytopes

$$
\begin{array}{ll}
f_0 \geq 5 & \\
f_3 \geq 5 & \\
f_{02} - 3f_2 \geq 0 & \text{(equality for 2-simplicial)} \\
f_{02} - 3f_1 \geq 0 & \text{(equality for 2-simple)} \\
f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0 & \text{(equality for elementary)} \\
6f_1 - 6f_0 - f_{02} \geq 0 &
\end{array}
$$

More new results on 4-polytopes

Ling 2006

new nonlinear inequalities for flag vectors

$$(k-1)f_{02} - \binom{k+1}{2} f_2 + f_1 \leq \binom{f_0}{2}$$

$$2(k-1)f_{02} - k(k+1)f_2 + (k^2-3k+4)f_1 - k(k-3)f_0 \leq 4\binom{f_0}{2}$$

Difficulty

We do not know how to generate random combinatorial types of polytopes.

Note that the convex hull of a random set of points in $\mathbf{R}^d$ is a simplicial polytope.

This makes it difficult to test conjectures, and even to come up with conjectures.

Further work—specialization

Special classes of polytopes

- cubical (Adin, Babson, G. Blind, R. Blind, Chan, Hetyei, Jockusch, Joswig, Liu, Ziegler)
- k-simplicial, h-simple (Kalai, Paffenholz, Stenson, Werner, Ziegler)
- polytopes with symmetry (A'Campo-Neuen, Adin, Björner, Jorge, Novik, Stanley)
- zonotopes and geometric lattices (B, Billera, Ehrenborg, Kung, Nyman, Readdy, Stenson, Sturmfels, Swartz)
- 0/1 polytopes (Aichholzer, Bárány, Gatzouras, Giannopoulos, Kaibel, Markoulakis, Pór, Ziegler)
- cyclic-like polytopes (B, Bisztriczky, Dinh, Smilansky)

Cyclic-like polytopes

Generalizations of the simplex

multiplex
braxtope

Gale polytopes

facets satisfy "Gale's evenness condition"

Generalizations of cyclic polytopes

simplicial and Gale . . . . cyclic polytope

multiplicial and Gale . . . . ordinary polytope

braxial and Gale . . . . periodically cyclic Gale polytope

Further work—generalization

More general classes of partially ordered sets

- general graded posets (Billera, Hetyei, Liu)
- Eulerian posets (B, Billera, Chen, Ehrenborg, Hetyei, Jojić, Lau, Readdy, Reading, Stanley)
- Eulerian manifolds (Björner, Charney, Chen, Davis, Hersh, Kalai, Novik, Sparla, Yan)
- Gorenstein* lattices (Billera, Ehrenborg, Karu, Masuda, Murai, Readdy, Reading, Stanley)

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Connections with other mathematical structures

- toric varieties (Bressler, Bukhshtaber, Karu, Leung, Lunts, Panov, Reiner, Stanley)
- coalgebras (Ehrenborg and Readdy)
- Hopf algebra of quasisymmetric functions (Aguiar, N. Bergeron, Billera, Hsiao, Sottile, van Willigenburg)

The end

THANK YOU!

http://www.math.ku.edu/~bayer