Combinatorial Aspects of Convex Polytopes

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1 Definitions and Fundamental Results

1.1 Introduction

A *convex polyhedron* is a subset of ^{d} that is the intersection of a finite number of closed halfspaces. A bounded convex polyhedron is called a *convex polytope*. Since most polyhedra under consideration will be convex, this adjective will usually be omitted. Space limitations prevent a comprehensive survey of the entire theory of polytopes; therefore, this chapter will concentrate primarily upon some techniques that have been successful in analyzing their combinatorial properties. Other aspects of the role of polytopes in convexity are treated in some of the other chapters of this volume.

The following may be regarded as the fundamental theorem of convex polytopes.

Theorem 1.1 *$P \subset {}^d$ is a polytope if and only if it is the convex hull of a finite set of points in ^{d}.*

The problem of developing algorithms to convert from one description of a polytope to the other arises in mathematical programming and computational geometry. The above theorem and related results are foundational to the theory of linear programming duality, and one of the central themes of combinatorial optimization is to make this conversion for special polytopes related to specific programming problems. See, for example, Edelsbrunner [1987], Preparata-Shamos [1985], and the chapters *Geometric Aspects of Mathematical Programming* by Gritzmann and Klee and *Convexity in Discrete Optimization* by Burkard in this volume.

1.2 Faces

The *dimension*, $\dim P$, of a polyhedron P is the dimension of its affine span, and a k-dimensional polyhedron is called a *k-polyhedron* for brevity. The *faces* of P are , P, and the intersections of P with its supporting hyperplanes. The empty set and P itself are *improper faces*; the other faces are *proper*. Each face of P is itself a polyhedron, and a face of dimension j is called a *j-face*. If $\dim P = d$, faces of P of dimension 0, 1, $d-2$ and $d-1$ are called *vertices*, *edges*, *subfacets* (or *ridges*), and *facets*, respectively. A polytope equals the convex hull of its vertices. The *f-vector* of P is the vector $f = (f_0, f_1, \ldots, f_{d-1})$, where $f_j = f_j(P)$ denotes the number of j-faces of P.

Theorem 1.2 *The collection of all the faces of a polyhedron P, ordered by inclusion, is a lattice.*

This lattice is called the *face lattice* or *boundary complex* of P, and two polytopes are (*combinatorially*) *equivalent* if their face lattices are isomorphic.

Suppose $F \subset G$ are two distinct faces of a polytope P. Then the interval $[F, G]$ is isomorphic to the face lattice of some polytope Q, called a *quotient polytope* of P. In the case that $G = P$, we write $Q = P/F$.

1.3 Polarity and Duality

Suppose $P \subset \ ^d$ is a d-polytope containing the origin o in its interior. Then $P^* = \{x \in \ ^d : \langle x, y \rangle \leq 1 \text{ for all } y \in P\}$ is also a d-polytope, called the *polar* of P (with respect to o).

Theorem 1.3 *The face lattices of P and P^* are anti-isomorphic.*

Two polytopes with anti-isomorphic face lattices are said to be *dual*. Two important dual classes of d-polytopes are the class of *simplicial* d-polytopes, those for which every proper face is a simplex, and the class of *simple* d-polytopes, those for which every vertex is incident to exactly d edges.

1.4 Overview

Our survey begins with a discussion of shellability (Section 2), an influential notion which links early results in polytopes to some of the most important recent achievements. In Section 3 we discuss the powerful tools in commutative algebra and algebraic geometry which have so successfully and dramatically enriched the theory of polytopes; see also the chapter *Algebraic Geometry and Convexity* by Ewald in this volume. Gale transforms, another early tool, and their relationship to the blossoming theory of oriented matroids, are treated in Section 4 (see also the chapter *Oriented Matroids* by Bokowski in this volume). Section 5 considers problems centered around the graphs (1-skeletons) of polytopes, and we conclude with Section 6 which discusses some issues of realizability and combinatorial types.

The standard reference for the foundation of the theory of polytopes and results through 1967 is the influential book by Grünbaum [1967]. McMullen-Shephard [1971] and Brøndsted [1983] are briefer introductions that also contain more information on face numbers. For more on regular polytopes, see Coxeter [1963]. Klee and Kleinschmidt [1991] give a comprehensive survey of results in the combinatorial structure of polytopes.

2 Shellings

2.1 Introduction

A *shelling* of the boundary complex of a polytope is an ordering $F_1, F_2, \ldots, F_n$ of its facets such that $F_j \cap \bigcup_{i=1}^{j-1} F_i$ is homeomorphic to a $(d-2)$-dimensional ball or sphere for all j, $2 \leq j \leq n$. Many early "proofs" of Euler's relation exploited the intuitively appealing and seemingly obvious property that every polytope is shellable (see Grünbaum [1967]), but this was not established until 1971, and until then examples of nonshellable simplicial 3-balls had suggested that in fact it might be false. See also Danaraj-Klee [1974], McMullen-Shephard [1971], and Stillwell [1980].

Theorem 2.1 (Bruggesser-Mani [1971]) *The facets of any d-polytope P can be ordered $F_1, F_2, \ldots, F_n$ such that for all j, $2 \leq j \leq n-1$, $F_j \cap \bigcup_{i=1}^{j-1} F_i$ is the union of the first k facets of F_j in some shelling of F_j, for some k, $0 < k < f_{d-2}(F_j)$.*

2.2 Euler's Relation

Euler's relation is the generalization of the familiar equation $f_0 - f_1 + f_2 = 2$ for 3-polytopes, and provides a necessary condition for the f-vector.

Theorem 2.2 (Euler's Relation, Poincaré [1893, 1899]) *If P is a d-polytope, then*

$$\sum_{j=0}^{d-1} (-1)^j f_j = 1 - (-1)^d.$$

Moreover, this is the only affine relation satisfied by all f-vectors of d-polytopes.

Refer to Grünbaum [1967] for the history of this result. Though there now exist elementary combinatorial proofs of Euler's relation, the fact that the first real proof, by Poincaré, involved algebraic techniques, foreshadowed the recent fruitful interaction among polytopes, commutative algebra, and algebraic geometry.

In three dimensions, Euler's relation with some simple inequalities characterizes f-vectors of 3-polytopes.

Theorem 2.3 (Steinitz [1906]) *A vector (f_0, f_1, f_2) of nonnegative integers is the f-vector of a 3-polytope if and only if the following three conditions hold.*

i. $f_1 = f_0 + f_2 - 2$.

ii. $4 \leq f_0 \leq 2f_2 - 4$.

iii. $4 \leq f_2 \leq 2f_0 - 4$.

On the other hand, for no $d \geq 4$ has the set of all f-vectors of d-polytopes been completely characterized, though considerable progress has been made in the case $d = 4$—see Section 3.8.

2.3 Line Shellings

To sketch the proof that a polytope P is shellable, without loss of generality assume $P = \{x \in \ {}^d : \langle a_i, x\rangle \leq 1 \text{ for all } i,\ 1 \leq i \leq n\}$, with a_i normal to facet F_i. Choose vector $c \in \ {}^d$ such that the inner products $\langle c, a_i\rangle$ are all distinct. Relabel the facets, if necessary, so that $\langle c, a_1\rangle > \langle c, a_2\rangle > \cdots > \langle c, a_n\rangle$. Then $F_1, F_2, \ldots, F_n$ is a shelling order, and is called a *line shelling.* Geometrically, one begins at the origin, travels along a line L in the direction c, and lists the facets of P in the order in which they become visible, i.e., in the order in which the corresponding supporting hyperplanes are crossed. Then one returns from the opposite direction, listing the facets on the other side of P in the order in which they disappear from view. This idea can be generalized to *curve shellings* in which one travels along an appropriate curve instead of a straight line.

That not all shellings are curve shellings is perhaps believable. In fact Smilansky [1988] proves that there exist shellings of some d-polytope P such that for no polytope Q combinatorially equivalent to P are the corresponding shellings of Q curve shellings.

By exploiting line shellings, Seidel [1986] obtains an algorithm to compute the convex hull of a finite set of points in affinely general position in logarithmic cost per face.

2.4 Shellable Simplicial Complexes

The definition of shellability can be extended to more general complexes. A *simplicial complex* Δ is a nonempty collection of subsets of a finite set V such that $G \in \Delta$ whenever $G \subseteq F$ for some $F \in \Delta$. Members of Δ are its *faces* and the *dimension* $\dim F$ of a face F is $|F| - 1$. The *dimension* of Δ, $\dim \Delta$, is $\max\{\dim F : F \in \Delta\}$. So the boundary complex of a simplicial d-polytope is a simplicial $(d-1)$-complex. As with polytopes, faces of a simplicial $(d-1)$-complex of dimension 0, 1, $d-2$ and $d-1$ are called *vertices*, *edges*, *subfacets* (or *ridges*), and *facets* of Δ, respectively.

To say Δ is *shellable* means that all of its maximal faces are facets, and the facets can be ordered $F_1, F_2, \ldots, F_n$ such that

$$F_m \cap \bigcup_{j=1}^{m-1} F_j = \bigcup_{j=1}^{k_m} G_{mj} \text{ for all } m,\ 2 \leq m \leq n,$$

where $G_{m1}, G_{m2}, \ldots, G_{mk_m}$ are $k_m > 0$ distinct subfacets of Δ (facets of F_m) for all m, $2 \leq m \leq n$. Taking $k_0 = 0$, the numbers k_m readily determine the f-vector of Δ. For, let $h_i = \text{card}\{m : k_m = i\}$. Then McMullen and Walkup [1971] show

$$f_j = \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_i \text{ for all } j,\ -1 \leq j \leq d-1. \tag{1}$$

Because these equations are invertible,

$$h_i = \sum_{j=0}^{i} (-1)^{j-i} \binom{d-j}{d-i} f_{j-1} \text{ for all } i,\ 0 \le i \le d, \tag{2}$$

the quantities h_i are independent of the shelling order, and can in fact be *defined* via Equation (2), even for nonshellable simplicial complexes or for more general collections of subsets of a finite set. The vector $h = (h_0, h_1, \ldots, h_d)$ is the *h-vector* of Δ and contains the same information as the f-vector. The above discussion shows that h-vectors of shellable simplicial complexes are nonnegative.

2.5 The Dehn-Sommerville Equations

Let P be a simplicial d-polytope containing o in its interior, and take a line shelling $F_1, F_2, \ldots, F_n$ of P induced by a direction c. Then the vector $-c$ induces the shelling $F_n, F_{n-1}, \ldots, F_1$, showing that the shelling is reversible. Since each subfacet is contained in exactly two facets, if facet F_m contributes to h_i in the first shelling, then it contributes to h_{d-i} in the second. As a consequence of the invariance of the h-vector, it must be symmetric.

Theorem 2.4 (Dehn-Sommerville Equations, Sommerville [1927]) *For a simplicial d-polytope,*

$$h_i = h_{d-i} \textit{ for all } i,\ 0 \le i \le d.$$

Equivalently,

$$f_i = \sum_{j=i}^{d-1} (-1)^{d-j-1} \binom{j+1}{i+1} f_j \textit{ for all } i,\ -1 \le i \le d-2.$$

Moreover, any affine relation satisfied by all f-vectors of simplicial d-polytopes is an affine combination of the above equations.

Hence the affine span of the set of all f-vectors of simplicial polytopes has dimension $\lfloor d/2 \rfloor$. Note that $1 = h_0 = h_d$ is equivalent to Euler's relation. The transformation of the f-vector into the h-vector and the above formulation of the Dehn-Sommerville equations in terms of the h-vector was already known to Sommerville, although he was not aware of the algebraic interpretation of the h-vector (discussed in Section 3).

It is an easy matter to verify that if Δ is a shellable simplicial $(d-1)$-complex such that every subfacet is contained in exactly two facets, then such a complex must be a p.l.-sphere and every shelling of Δ is reversible. So it is easy to see that the Dehn-Sommerville equations hold for shellable spheres as well. In fact, the first proofs of the Dehn-Sommerville equations did not depend upon shellability and show that these equations hold for homological $(d-1)$-spheres as well as for some more general simplicial complexes. See Grünbaum [1967].

Although f-vectors of simplicial polytopes are not always unimodal, some inequalities are a consequence of the Dehn-Sommerville equations.

Theorem 2.5 (Björner [1981]) *The f-vector of a simplicial $(d-1)$-sphere satisfies*

$$f_0 < f_1 < \cdots < f_{\lfloor d/2 \rfloor - 1} \leq f_{\lfloor d/2 \rfloor}$$

and

$$f_{\lfloor 3(d-1)/4 \rfloor} > \cdots > f_{d-2} > f_{d-1}.$$

It turns out, however, that the *h-vectors* of simplicial polytopes are unimodal—see Section 3.

The equations imply that the f-vector of the boundary of a triangulated ball is determined by the f-vector of the ball itself (Section 4.3). Let Q be any unbounded simple d-polyhedron with at least one vertex. Then there exists a simplicial d-polytope P with a vertex v such that lattice of the nonempty faces of Q is anti-isomorphic to the lattice of faces of P that do not contain v. The next result is a consequence of this duality, and is mentioned in Billera-Lee [1981a].

Theorem 2.6 *If Q is a simple d-polyhedron with at least one vertex, then the number of unbounded k-faces of Q equals*

$$f_k - \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} f_j, \ \ 1 \leq k \leq d-1.$$

2.6 Completely Unimodal Numberings and Orientations

Returning to P and the line shelling induced by c as in Section 2.3, let Q be the simple d-polytope that is the polar of P. Any given acyclic orientation of the edges of Q and any given numbering of the vertices of Q from 1 to n are said to be *consistent* provided the edges are oriented from lower-numbered vertices to higher-numbered vertices. Associated with any numbering is a unique consistent orientation, and associated with any acyclic orientation is at least one consistent numbering.

Our indexing of the facets of P implies that $\langle c, a_i \rangle > \langle c, a_j \rangle$ if and only if $i < j$. Label the vertex a_i of Q with the number i, $i = 1, \ldots, n$, and consider the associated consistent orientation. Then h_i equals the number of vertices of Q having in-degree i, as well as the number of vertices having out-degree i.

This numbering of vertices also has the property that for every k-face of Q, $2 \leq k \leq d$, the restriction of the associated consistent orientation to that face has a unique vertex of in-degree zero. Any numbering of the vertices of Q

possessing the above property will be called *completely unimodal*, and an acyclic orientation is said to be *completely unimodal* if any (equivalently, all) consistent numberings are completely unimodal. Such numberings and orientations may be regarded as abstract objective functions. See also Brøndsted [1983].

Theorem 2.7 (Williamson Hoke [1988]) *The following are equivalent for a given numbering of the vertices of Q:*

i. The numbering is completely unimodal.

ii. h_i equals the number of vertices having in-degree (out-degree) i in the associated consistent orientation.

iii. For all k, $1 \leq k \leq n$, and any face F, the edge-graph induced by the set of vertices on F numbered less than (greater than) k is connected.

iv. The induced ordering of the facets of P is a (not necessarily curve) shelling.

v. In every k-face of P, $2 \leq k \leq d$, there is a unique vertex of in-degree (out-degree) zero with respect to the induced consistent orientation of the edges of that face.

Kalai characterizes completely unimodal orientations in the case that the edge-graph of a simple d-polytope Q is given, but otherwise its facial structure is unknown. For any acyclic orientation O of the graph, let h_k^O be the number of vertices with in-degree k. Define $f^O = h_0^O + 2h_1^O + 4h_2^O + \cdots + 2^d h_d^O$.

Theorem 2.8 (Kalai [1988c]) *The following are equivalent for an acyclic orientation O^*.*

i. O^* is completely unimodal.

ii. f^{O^*} minimizes f^O over all acyclic orientations.

iii. f^{O^*} equals the total number of nonempty faces of Q.

From this Kalai obtains a new proof of a result first established by Blind and Mani-Levitska.

Theorem 2.9 (Blind - Mani-Levitska [1987]) *The edge-graph of a simple polytope completely determines its entire combinatorial structure.*

2.7 The Upper Bound Theorem

Given integers $d \geq 2$ and $n \geq d+1$, take the convex hull of any n distinct points on the moment curve $(t, t^2, \ldots, t^d)$. The combinatorial structure of the resulting simplicial d-polytope $C(n,d)$ is independent of the actual choice of points, and this polytope is referred to as the *cyclic d-polytope with n vertices.* It turns out that every subset of $k \leq d/2$ vertices of $C(n,d)$ forms a face ($C(n,d)$ is *neighborly*), so it was conjectured that this polytope has the largest number of faces of all dimensions of all convex d-polytopes with n vertices. Explicit formulas for $f_j(C(n,d))$ can be found in Brøndsted [1983], Grünbaum [1967], or McMullen-Shephard [1971]; we mention that

$$f_{d-1}(C(n,d)) = \binom{n - \lfloor \frac{1}{2}(d+1) \rfloor}{n-d} + \binom{n - \lfloor \frac{1}{2}(d+2) \rfloor}{n-d}.$$

Theorem 2.10 (Upper Bound Theorem, McMullen [1970]) *Let P be a d-polytope with n vertices. Then $f_j(P) \leq f_j(C(n,d))$ for all j, $1 \leq j \leq d-1$.*

A perturbation argument shows that it suffices to prove this result for simplicial d-polytopes. McMullen uses properties of line shellings to show that $h_i(P) \leq \binom{n-d+i-1}{i}$ for all i, $0 \leq i \leq d$. But the fact that $h_i(C(n,d)) = \binom{n-d+i-1}{i}$ for all i, $0 \leq i \leq \lfloor d/2 \rfloor$, together with the Dehn-Sommerville equations, imply that $h_i(P) \leq h_i(C(n,d))$ for all i, $0 \leq i \leq d$. The result now follows immediately from the observation that the f_j are nonnegative combinations of the h_i. See McMullen-Shephard [1971] for an account of the solution of the Upper Bound Conjecture. The proof can also be found in Brøndsted [1983].

2.8 The Lower Bound Theorem

Starting with a d-simplex, one can add new vertices by building shallow pyramids over facets to obtain a simplicial convex d-polytope with n vertices, called a *stacked* polytope. If $P(n,d)$ is such a polytope, then

$$f_j = \begin{cases} \binom{d}{j} n - \binom{d+1}{j+1} j, & \text{if } 0 \leq j \leq d-2, \\ (d-1)n - (d+1)(d-2), & \text{if } j = d-1. \end{cases}$$

It was conjectured that no simplicial d-polytope with n vertices can have fewer faces than $P(n,d)$, and a certain reduction implied that it was sufficient to show this result for f_1. Barnette proved this conjecture about the same time that McMullen established the Upper Bound Theorem. Barnette's argument does not use the full strength of shellability, but relies upon a weaker ordering of the facets of the dual simple polytope. The proof also appears in Brøndsted [1983].

Theorem 2.11 (Lower Bound Theorem, Barnette [1971, 1973]) *Let P be a simplicial d-polytope with n vertices and $P(n,d)$ be a stacked d-polytope with n vertices. Then*

$$f_j(P) \geq f_j(P(n,d)) \text{ for all } j,\ 1 \leq j \leq d-1. \tag{3}$$

Moreover, if $d \geq 4$ and equality occurs in (3) for any one value of j, then P must itself be stacked.

The case of equality for $j = d-1$ was proved by Barnette and for the remaining values of j by Billera and Lee [1981b]. Connections between the Lower Bound Theorem and rigidity were discovered by Kalai and will be discussed in Section 3.13.

Another type of lower bound theorem was proved by Blind and Blind. Write C^d for the d-cube.

Theorem 2.12 (Blind-Blind [1990]) *Let P be a d-polytope with no triangular faces. Then*

$$f_j(P) \geq f_j(C^d) \text{ for all } j,\ 0 \leq j \leq d-1. \tag{4}$$

Moreover, if equality occurs in (4) for any one value of j, then P must itself be a d-cube.

2.9 Constructions Using Shellings

Given any two positive integers h and i, there is a unique sequence of integers $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$ such that

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}.$$

The *ith pseudopower of h* is then defined as

$$h^{<i>} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}.$$

For convenience we define $0^{<i>}$ to be 0 for any positive integer i.

Stanley characterized the h-vectors (and hence the f-vectors) of shellable simplicial complexes.

Theorem 2.13 (Stanley [1977]) *A vector $h = (h_0, h_1, \ldots, h_d)$ of nonnegative integers is the h-vector of some shellable simplicial $(d-1)$-complex if and only if $h_0 = 1$ and $h_{i+1} \leq h_i^{<i>}$ for all i, $1 \leq i \leq d-1$.*

The algebraic methods that imply the necessity of these conditions are discussed in Section 3.2. But the sufficiency is much more straightforward. Given h satisfying the pseudopower conditions, let $n = h_1 + d$ and $V = \{1, 2, \ldots, n\}$. Take $\mathcal{F}$ to be the collection of all subsets of V of cardinality d, and $\mathcal{F}_i$ be those members F of $\mathcal{F}$ such that $d + 1 - i$ is the smallest element of V not in F. For two members F and G of $\mathcal{F}$, say $F < G$ if the largest element of V in their symmetric difference is in G. For each i, $0 \le i \le d$, choose the first (in the given ordering) h_i members of $\mathcal{F}_i$. The resulting collection $\mathcal{C}$ consists of the facets of the desired shellable complex, and the given ordering induces the shelling order. This result and the accompanying construction are reminiscent of the characterization of the f-vectors of arbitrary simplicial complexes by Kruskal [1963] and Katona [1968].

Now suppose we have a vector $h = (h_0, h_1, \ldots, h_d)$ of nonnegative integers such that $h_i = h_{d-i}$ for all i, $1 \le i \le d$, and $h_{i+1} - h_i \le (h_i - h_{i-1})^{<i>}$ for all i, $1 \le i \le \lfloor d/2 - 1 \rfloor$. These are known as the McMullen conditions; see Section 3.4. Billera and Lee [1981b] show how to extend the above construction to obtain a shellable d-ball whose boundary is a simplicial $(d-1)$-sphere having h-vector h. Set $n = h_1 + d$, $V = \{1, 2, \ldots, n\}$, and regard V as the set of vertices of a cyclic $(d+1)$-polytope $C(n, d+1)$ with n vertices. For all i, $1 \le i \le \lfloor d/2 \rfloor$, define $\mathcal{F}_i$ to be the collection of all subsets F of V of cardinality $d+1$ corresponding to facets of the cyclic polytope $C(n, d+1)$ such that $d+2-2i$ is the smallest element of V not in F. For two members F and G of $\mathcal{F}_i$, say $F < G$ if the largest element of V in their symmetric difference is in G. For each i, $0 \le i \le \lfloor d/2 \rfloor$, choose the first (in the given ordering) $h_i - h_{i-1}$ members of $\mathcal{F}_i$ (with the convention that $h_{-1} = 0$). The resulting collection $\mathcal{C}$ consists of the facets of a d-ball Δ which are shellable in the given ordering. Further, $h(\Delta) = (h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1}, 0, 0, \ldots, 0)$, whence the h-vector of the boundary $\partial\Delta$ of Δ equals $(h_0, h_1, \ldots, h_d)$ (see Equation (4.9) of Section 4.3). In fact, $\partial\Delta$ can be realized as the boundary of a simplicial convex d-polytope; see Section 3.4.

Kalai generalizes this construction to prove that there are many simplicial spheres.

Theorem 2.14 (Kalai [1988a]) *For fixed d, the number of combinatorial types of triangulated $(d-1)$-spheres with n vertices is between $e^{bn^{\lfloor (d-1)/2 \rfloor}}$ and $n^{cn^{\lfloor d/2 \rfloor}}$, where b and c are positive constants.*

A comparison with Theorem 6.3 shows that there are many more simplicial spheres than polytopes.

2.10 Notes

Geometric analogues of Euler's relation and the Dehn-Sommerville equations can be found in Grünbaum [1967]. Lawrence [1991] uses relatives of these to show that the volume of a polytope described by rational inequalities is not polynomially expressible in terms of the describing data.

The class of boundary complexes of simplicial polytopes is contained in the class of shellable spheres, which is contained in the class of p.l.-spheres, which is contained in the class of topological spheres, which is contained in the class of homological spheres. All inclusions are proper. For example, there exist shellable spheres that are not polytopal (Danaraj-Klee [1978b]), and the nondecidability result of Volojin, Kuznetsov and Fomenko [1974] for determining whether or not a given complex is a p.l.-sphere implies that nonshellable p.l.-spheres must exist (see Mandel [1982]).

In two dimensions the situation is somewhat simpler; see Danaraj-Klee [1978a], Grünbaum [1967], and Section 5.2.

Theorem 2.15 *The following conditions are equivalent for a 2-complex.*

i. The complex is polytopal.

ii. The complex is a sphere.

iii. The complex is a shellable closed pseudomanifold.

In three dimensions, all simplicial 3-spheres with at most 9 vertices are shellable (Danaraj-Klee [1978b]).

3 Algebraic Methods

3.1 Introduction

In this section we explore the developing relationship between techniques in commutative algebra and algebraic geometry and results in the combinatorial structure of polytopes. This interaction was launched by Stanley's use of the Stanley-Reisner ring to extend the Upper Bound Theorem to simplicial spheres, and was further propelled by Stanley's short and dramatic proof of the McMullen conditions based upon connections between the Stanley-Reisner ring of a polytope and the cohomology of an associated toric variety.

3.2 The Stanley-Reisner Ring

The Stanley-Reisner ring of a simplicial complex encodes the simplices of the complex as monomials. Reisner's Theorem allows a translation of topological properties of the complex into algebraic properties of the ring. In particular, the Stanley-Reisner ring of the boundary complex of a simplicial convex polytope is Cohen-Macaulay. This is what enabled Stanley to prove the Upper Bound Theorem for simplicial spheres. He also used it in his proof of the necessity of the McMullen conditions. In what follows k is a fixed infinite field.

Let Δ be a simplicial complex with vertices $v_1, v_2, \ldots, v_n$, each simplex (face) being identified with its set of vertices. In the polynomial ring $k[x_1, x_2, \ldots, x_n]$, let I_Δ be the ideal generated by all monomials $x_{i_1} x_{i_2} \cdots x_{i_s}$ such that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\} \notin \Delta$. The *Stanley-Reisner ring* (or *face ring*) of Δ is the quotient ring $R_\Delta = k[x_1, x_2, \ldots, x_n]/I_\Delta$. See Stanley [1975].

Let R_m be the vector subspace of R_Δ generated by the monomials of degree m in R_Δ. This gives a grading of the Stanley-Reisner ring, $R_\Delta = \bigoplus_{m \geq 0} R_m$. As a k-algebra R_Δ is generated by the monomials of R_1, that is, by the variables $x_1, x_2, \ldots, x_n$ themselves. The graded component R_m is spanned by the degree m monomials whose supports are in the complex. The number of monomials of degree m with given support depends only on the size of the support. Thus we can write the Hilbert function of the graded algebra in terms of the f-vector of the complex. The *Hilbert function* of the graded algebra R_Δ is the function $H : \quad \to \quad$ given by $H_m = H(R_\Delta, m) = \dim_k R_m$. If Δ is a simplicial complex of dimension $d-1$ (for example, the boundary complex of a simplicial d-polytope) then (Stanley [1975])

$$H(R_\Delta, m) = \begin{cases} 1, & \text{if } m = 0, \\ \displaystyle\sum_{i=0}^{d-1} \binom{m-1}{i} f_i, & \text{if } m > 0. \end{cases} \tag{5}$$

Macaulay essentially gives a numerical characterization of the Hilbert functions of graded algebras generated by their degree 1 elements. Recall the definition of pseudopower given in Section 2.9.

Theorem 3.1 (Macaulay [1927]) *$H_0, H_1, H_2, \ldots$ is the Hilbert function of a graded algebra generated by degree 1 elements if and only if $H_0 = 1$ and, for $m > 0$, $0 \leq H_{m+1} \leq H_m^{<m>}$.*

We could apply Macaulay's theorem to the Hilbert function of the Stanley-Reisner ring to get inequalities on the f-vectors of simplicial complexes. We are interested primarily in polytopes (or spheres), and in this case the inequalities say little. Instead we wish to apply Macaulay's theorem to a quotient of the Stanley-Reisner ring.

A graded ring is called *Cohen-Macaulay* if its Krull dimension equals its depth. We do not have the space to elaborate on this; for more information see Stanley [1975]. We note only the following property of Cohen-Macaulay rings. If a graded k-algebra R of the above form is Cohen-Macaulay of dimension d, then R has linear (degree 1) elements $\theta_1, \theta_2, \ldots, \theta_d$ such that R is a finitely generated, free module over $k[\theta_1, \theta_2, \ldots, \theta_d]$. In this case the Hilbert function h of the quotient algebra $R/(\theta_1, \theta_2, \ldots, \theta_d)$ is related to the Hilbert function H of R by the following simple relationship:

$$\sum_{i=0}^{d} h_i t^i = (1-t)^d \sum_{i=0}^{\infty} H_i t^i.$$

We apply this to the Stanley-Reisner ring of a simplicial sphere. The theorem of Reisner [1976] gives a topological criterion for the face ring of a simplicial complex to be Cohen-Macaulay. In particular, the Stanley-Reisner ring of a sphere is Cohen-Macaulay. The same is true for a shellable complex; see Kind-Kleinschmidt [1979] for a more elementary proof. The Hilbert function of the quotient algebra $R/(\theta_1, \theta_2, \ldots, \theta_d)$, like that of R itself, can be expressed in terms of the f-vector of the simplicial complex. The Hilbert function of the quotient turns out to be the h-vector, and in fact is given by Equation (2) of Section 2.4. Recall that the Dehn-Sommerville equations for simplicial convex polytopes have the simple form $h_i = h_{d-i}$.

Now Macaulay's theorem can be applied to the quotient algebra $R/(\theta_1, \theta_2, \ldots, \theta_d)$, whose Hilbert function is the h-vector. This gives the relations $h_0 = 1$ and, for $m > 0$, $0 \leq h_{m+1} \leq h_m^{<m>}$.

These inequalities imply the following: for a simplicial d-polytope and for $0 \leq i \leq d/2$,

$$h_i \leq \binom{n-d+i-1}{i}.$$

These inequalities imply the Upper Bound Theorem for simplicial spheres. The Upper Bound Theorem was first proved by McMullen for simplicial polytopes (see Section 2.7). It was proved for arbitrary simplicial spheres by Stanley [1975] using the method outlined here.

3.3 Toric Varieties

After he introduced the face ring of a simplicial complex and proved the Upper Bound Theorem for simplicial spheres, Stanley learned of a connection with algebraic geometry.

A certain type of algebraic variety, a projective toric variety, comes equipped with a moment map into real Euclidean space. The image of this map is a rational convex polytope (rationality here refers to the coordinates of the vertices). When the toric variety has only relatively mild singularities, the corresponding polytope is simplicial. From the combinatorial viewpoint, rationality places no restriction on *simplicial* polytopes—every simplicial polytope is combinatorially equivalent to a rational polytope. This is not the case for nonsimplicial polytopes—there are combinatorial types of polytopes not realized by any rational polytope (see Sections 4.2 and 6.5).

The toric variety can be described explicitly in terms of the convex polytope. See Fine [1985] and the chapter *Algebraic Geometry and Convexity* by Ewald in this volume. Let P be a convex d-polytope with vertex set $V = \{v_1, v_2, \ldots, v_n\} \subset {}^d$. Any affine dependence on V with integer coefficients can be written in the form $\sum_{i=1}^n b_i v_i = \sum_{i=1}^n c_i v_i$, where for all i, $b_i, c_i \in$ and $\sum_{i=1}^n b_i = \sum_{i=1}^n c_i$. Let $\mathcal{A}_P$ be the set of pairs (b, c) of coefficient vectors arising in this manner. Using the notation $x^b = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$, with the convention

$0^0 = 1$, define E_P to be

$$E_P = \{x \in \ {}^n : x^b = x^c \text{ for all } (b, c) \in \mathcal{A}_P\}.$$

Although $\mathcal{A}_P$ is an infinite set, for a rational polytope P it is a finitely generated semigroup. Thus E_P is a subset of $\ {}^n$ defined by a finite number of polynomial equations, hence is an algebraic variety. Finally E_P is "projectivized": we define T_P to be $E_P \setminus \{0\}$ modulo the relation $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \ {}^*$. This is the toric variety associated with the polytope P (though not precisely the variety discussed by Danilov [1978]).

The cohomology ring of T_P for simplicial (rational) polytopes P is computed by Danilov [1978]. It turns out to be isomorphic to the quotient $A_\Delta = R_\Delta/(\theta_1, \theta_2, \ldots, \theta_d) = \bigoplus_{m \geq 0} A_m$ of the Stanley-Reisner ring modulo a certain linear system of parameters (here Δ is the boundary complex of P). Later Saito [1985] proved that the Hard Lefschetz Theorem holds for the varieties T_P (P simplicial). This ensures the existence of an element $\omega \in A_1$, called a *hyperplane section,* such that the maps $A_i \stackrel{\times\omega}{\to} A_{i+1}$ are injective for all i, $0 \leq i \leq (d-1)/2$. Thus A_Δ/ω is a graded algebra with Hilbert function $H(A_\Delta/\omega, m) = h_m - h_{m-1}$ for $0 \leq m \leq d/2$.

3.4 The McMullen Conditions

We are now able to state the major theorem, conjectured by McMullen [1971] and proved by Billera and Lee [1981b] and Stanley [1980b].

Theorem 3.2 (The McMullen Conditions) *A vector* $(h_0, h_1, \ldots, h_d) \in \ {}^{d+1}$ *is the h-vector of a simplicial polytope if and only if*

i. $h_i = h_{d-i}$ *for all* i*,* $0 \leq i \leq d$*.*

ii. $h_0 = 1$*, and* $h_i \leq h_{i+1}$ *for all* i*,* $0 \leq i \leq d/2 - 1$*.*

iii. $h_{i+1} - h_i \leq (h_i - h_{i-1})^{<i>}$ *for all* i*,* $1 \leq i \leq d/2 - 1$*.*

Since the h-vector and f-vector are linearly equivalent, this theorem characterizes the f-vectors of simplicial polytopes. Of course, it also characterizes the f-vectors of simple polytopes, which are obtained by reversing the f-vectors of simplicial polytopes.

The "sufficiency" was proved constructively by Billera and Lee. They use a monomial algebra with Hilbert function $h_i - h_{i-1}$ to select a certain subset $\mathcal{C}$ of facets of the cyclic polytope $C(n, d+1)$ as described in Section 2.9. Then they show that by selecting points $(t_i, t_i^2, \ldots, t_i^{d+1})$ on the moment curve such that $t_1 \ll t_2 \ll \cdots \ll t_n$, one can place a new point v beyond precisely the facets in $\mathcal{C}$. The construction is completed by taking the convex hull Q of $C(n, d+1)$ and v and passing to the vertex figure of v (the intersection of Q with a hyperplane strictly separating v from the other vertices of Q).

The "necessity" of the McMullen conditions, i.e., that all h-vectors of simplicial polytopes satisfy (i–iii), was proved by Stanley using toric varieties, as outlined in Section 3.3. Note that condition (i) (the Dehn-Sommerville equations) reflects Poincaré duality of the cohomology of the toric variety, but we have seen much simpler proofs (Section 2.5).

Stanley's proof of the necessity of the McMullen conditions is unsatisfying for several reasons. One reason is that the Hard Lefschetz Theorem, which is crucial to the proof, lies well beyond combinatorics. The proof depends heavily on the specific geometry of the polytope's embedding, not just on its combinatorial structure. In particular, it does not prove the McMullen conditions for h-vectors of simplicial spheres that are not polytopal (see Section 6.5). McMullen himself did not know of the toric variety connection when he made his conjecture. His idea arose from consideration of the geometry of polytopes coded in Gale diagrams (see Section 4).

3.5 Polytope Pairs

The McMullen conditions provide a characterization of the f-vectors of simple polytopes, which arise naturally in optimization. In this context it is also natural to ask about unbounded polyhedra. Just as duality establishes a correspondence between simple polytopes and simplicial polytopes, so also is there a correspondence between unbounded simple polyhedra and simplicial polytope pairs.

A (*simple*) *polytope pair of type* (d, v, d', v') is a pair (P^*, F^*), where P^* is a simple convex d-polytope with v facets and F^* is a d'-face of P^* with v' facets.

Associated with a polytope pair (P^*, F^*) is an unbounded simple d-polyhedron Q^*, obtained by applying a projective transformation that sends a supporting hyperplane for F^* onto the hyperplane at infinity. The polyhedron Q^* has recession cone of dimension $d'+1$. Conversely, every simple, pointed convex d-polyhedron with $(d'+1)$-dimensional recession cone can be associated to some polytope pair of type (d, v, d', v'), for some v and v'.

Methods for facial enumeration in simplicial polytopes were used by Klee [1974], Billera and Lee [1981a], Lee [1984], and Barnette, Kleinschmidt, and Lee [1986] to develop bounds on the numbers of faces of polytope pairs. The pairs are first dualized.

Let $k = d-d'$ and $r = d-d'+v'$. The dual of a simple polytope pair (P^*, F^*) of type (d, v, d', v') is a *simplicial polytope pair* (P, F) of type (d, v, k, r), where P is a simplicial convex d-polytope with v vertices, and F is a $(k-1)$-face of P contained in $r-k$ k-faces of P. Let $\Gamma = \partial P \setminus F$, the simplicial complex obtained by deleting the face F (and all faces containing F) from the simplicial complex ∂P, the boundary of P. The faces of the boundary $\partial\Gamma$ of Γ correspond to the unbounded faces of Q^*. Thus to estimate the numbers of bounded and unbounded faces of Q^* we use estimates on the numbers of faces (or h-vectors) of P and Γ.

Theorem 3.3 (Upper Bound Theorem for Polytope Pairs) *Let* $3 \le d < r \le v$ *and* $1 \le k \le d-2$. *Put* $n = \lfloor d/2 \rfloor$. *Let* (P,F) *range over all (simplicial) polytope pairs of type* (d,v,k,r), *taking* $\Gamma = \partial P \setminus F$.

i. *If* $1 \le k \le (d-1)/2$ *then*

$$\max h_i(P) = \begin{cases} \binom{v-d+i-1}{i}, & \text{if } 0 \le i \le k, \\ \binom{v-d+i-1}{i} - \binom{v-d+i-k-1}{i-k} + \binom{r-d+i-k-1}{i-k}, & \text{if } k+1 \le i \le n, \\ \binom{v-i-1}{d-i} - \binom{v-i-k-1}{d-i-k} + \binom{r-i-k-1}{d-i-k}, & \text{if } n+1 \le i \le d-k-1, \\ \binom{v-i-1}{d-i}, & \text{if } d-k \le i \le d. \end{cases}$$

$$\max h_i(\Gamma) = \begin{cases} \binom{v-d+i-1}{i}, & \text{if } 0 \le i \le k-1, \\ \binom{v-d+i-1}{i} - \binom{v-d+i-k-1}{i-k}, & \text{if } k \le i \le n, \\ \binom{v-i-1}{d-i} - \binom{v-i-k-1}{d-i-k}, & \text{if } n+1 \le i \le d-k-1, \\ \binom{v-i-1}{d-i} - r + d, & \text{if } d-k \le i \le d-1, \\ 0, & \text{if } i = d. \end{cases}$$

ii. *If* $(d-1)/2 < k \le d-2$ *then*

$$\max h_i(P) = \begin{cases} \binom{v-d+i-1}{i}, & \text{if } 0 \le i \le n, \\ \binom{v-i-1}{d-1}, & \text{if } n+1 \le i \le d. \end{cases}$$

$$\max h_i(\Gamma) = \begin{cases} \binom{v-d+i-1}{i}, & \text{if } 0 \le i \le n, \\ \binom{v-i-1}{d-i}, & \text{if } n+1 \le i \le k-1, \\ \binom{v-i-1}{d-i} - 1, & \text{if } i = k, \\ \binom{v-i-1}{d-i} - r + d, & \text{if } k+1 \le i \le d-1, \\ 0, & \text{if } i = d. \end{cases}$$

Moreover, for each of the parts of (i) and (ii), the maxima are simultaneously achievable.

Theorem 3.4 (Lower Bound Theorem for Polytope Pairs) *Let* $4 \le d < r \le v$ *and* $2 \le k \le d-2$. *Put* $n = \lfloor d/2 \rfloor$ *and* $m = \lfloor (d-k)/2 \rfloor$. *Let* (P,F) *range over all (simplicial) polytope pairs of type* (d,v,k,r), *taking* $\Gamma = \partial P \setminus F$.

$$\min h_i(P) = \begin{cases} 1, & \text{if } i = 0, \\ v-d, & \text{if } 1 \le i \le n. \end{cases} \tag{6}$$

$$\min h_i(\Gamma) = \begin{cases} 1, & \text{if } i = 0, \\ v-d, & \text{if } 1 \le i \le k-1, \\ v-d-1, & \text{if } i = k, \\ v-r, & \text{if } k+1 \le i \le d-1, \\ 0, & \text{if } i = d. \end{cases} \tag{7}$$

For $k = 1$ and $3 \leq d < r \leq v$, equation (6) stays the same, but (7) becomes

$$\min h_i(\Gamma) = \begin{cases} 1, & \text{if } i = 0 \\ v - d - 1, & \text{if } i = 1 \\ v - r, & \text{if } 2 \leq i \leq d - 1 \\ 0, & \text{if } i = d. \end{cases}$$

All bounds can be achieved.

Theorems 3.3 and 3.4 are proved in Barnette, Kleinschmidt and Lee [1986] and Lee [1984], respectively. Theorem 3.3 applies as well to the case when P is an arbitrary simplicial sphere, but the lower bounds of Theorem 3.4 depend in an essential way upon the McMullen conditions.

3.6 Centrally Symmetric Simplicial Polytopes

A d-polytope P in ^{d} is *centrally symmetric* if for all points v in P, $-v$ is also in P. Björner conjectured (unpublished) that the h-vector of any centrally symmetric simplicial polytope satisfies the inequality $h_i - h_{i-1} \geq \binom{d}{i} - \binom{d}{i-1}$ for all i, $1 \leq i \leq d/2$. Stanley [1987b] proves this conjecture using the connection with toric varieties. This also proves lower bounds on the f-vectors of centrally symmetric simplicial polytopes, conjectured earlier by Bárány and Lovász [1982].

Any centrally symmetric simplicial d-polytope is combinatorially equivalent to a centrally symmetric simplicial d-polytope with rational vertices. The associated toric variety T_P and its cohomology ring inherit the action of the group of order 2 on the polytope P by virtue of central symmetry. Furthermore, this group action on the cohomology ring commutes with multiplication by the hyperplane section ω (see Section 3.3). So the cohomology ring decomposes as a direct sum of two graded vector spaces (one of which is a ring, the other of which is a module over this ring), on each of which multiplication by ω is injective. This gives Stanley's theorem.

Theorem 3.5 (Stanley [1987]) *If P is a centrally symmetric simplicial d-polytope, then*

$$h_i(P) - h_{i-1}(P) \geq \binom{d}{i} - \binom{d}{i-1} \quad \text{for all } i, 1 \leq i \leq d/2.$$

We summarize some consequences of this theorem.

Corollary 3.6 *Let P be a centrally symmetric simplicial d-polytope, and let $(h_0, h_1, \ldots, h_d)$ be its h-vector.*

i. $h_i \geq \binom{d}{i}$, *for all i, $0 \leq i \leq d$.*

ii. $f_i \geq 2^{i+1}\binom{d}{i+1} + 2(n-d)\binom{d}{i}$, *for all i, $0 \leq i \leq d-2$.*

iii. $f_{d-1} \geq 2^d + 2(n-d)(d-1)$.

iv. If for some i, $1 \leq i \leq d-1$, $h_i(P) = \binom{d}{i}$, *then* $h_j(P) = \binom{d}{j}$ *for all* j, *and* P *is affinely equivalent to a crosspolytope.*

Part (i) of the corollary was first conjectured by Björner; parts (ii) and (iii) are the conjecture of Bárány and Lovász.

3.7 Flag Vectors

In the previous sections of this chapter, the results on f-vectors of simplicial polytopes stemmed from interpretations of the h-vectors of the polytopes. One could define the h-vector of a nonsimplicial polytope by the same linear transformation of the f-vector, but none of the interpretations of h-vectors would continue to hold. In fact the vector so defined has negative components for some nonsimplicial polytopes. Furthermore, the f-vector captures much less of the combinatorial structure of a nonsimplicial than of a simplicial polytope. Thus in the study of arbitrary polytopes, attention has focused on other parameters.

The one that most directly generalizes the f-vector is the flag vector. Let P be a d-polytope. A chain of faces of P, $\subset F_1 \subset F_2 \subset \cdots \subset F_k \subset P$, is called an *$S$-flag*, where $S = \{\dim F_i \colon 1 \leq i \leq k\}$. The number of S-flags of P is denoted $f_S(P)$, and together these *flag numbers* form the *flag vector*, $(f_S(P))_{S \subseteq \{0,1,\ldots,d-1\}} \subseteq \ {}^{2^d}$. When writing a specific flag number we will drop the set brackets from the subscript. The f-vector is the projection of the flag vector onto the components with singleton indices. A flag number f_S of a d-simplex is a product of binomial coefficients depending only on d and S. Thus a flag number $f_S(P)$ of a simplicial d-polytope P depends only on the number of faces whose dimension is the largest element of S (and on d and S).

The problem of characterizing the f-vectors of polytopes extends to the problem of characterizing the flag vectors of polytopes. The main result on this problem is the specification of the affine hull of the flag vectors of polytopes of fixed dimension.

Theorem 3.7 (Bayer-Billera [1985]) *The affine dimension of the flag vectors of d-polytopes is $e_d - 1$, where (e_d) is the Fibonacci sequence, $e_d = e_{d-1} + e_{d-2}$, $e_0 = e_1 = 1$. The affine hull of the flag vectors is determined by the equations*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup\{j\}}(P) = \left(1 - (-1)^{k-i-1}\right) f_S(P),$$

where $i \leq k-2$, $i,k \in S \cup \{-1,d\}$, and S contains no integer between i and k.

These equations are called the *generalized Dehn-Sommerville equations*; their proof is similar to Sommerville's original proof of the Dehn-Sommerville equations for simplicial polytopes. For polytopes of dimension three, the generalized Dehn-Sommerville equations imply that the flag vector depends linearly on the f-vector. Thus Steinitz's characterization of the f-vectors of 3-polytopes (Theorem 2.3) extends to a characterization of the flag vectors of 3-polytopes.

Some inequalities are known to hold for the flag vectors of all polytopes. The most important of these was proved by Kalai using stress (see Section 3.13).

Theorem 3.8 (Kalai [1987]) *For all d-polytopes P*

$$f_{02}(P) - 3f_2(P) + f_1(P) - df_0(P) + \binom{d+1}{2} \geq 0.$$

The flag vector of a d-polytope P is a refinement of the f-vector of a simplicial d-polytope $\Delta(P)$, called the barycentric subdivision of P (see Section 6.6); the relationship is $f_i(\Delta(P)) = \sum_{|S|=i+1} f_S(P)$. The McMullen conditions applied to the barycentric subdivision thus give inequalities on the flag vector of the original polytope, but these are not sharp. The barycentric subdivision of a polytope is an example of a *completely balanced sphere*, studied in Stanley [1979]. There Stanley defined a refined or *extended h-vector* of a completely balanced sphere (see Sections 3.11 and 6.6). This extended h-vector is the Hilbert function of the Stanley-Reisner ring with respect to a fine grading. Unfortunately, no analogue of the Macaulay theorem (Theorem 3.1) is known for the extended h-vector. The extended h-vector of a shellable completely balanced sphere can also be calculated from a shelling.

3.8 Dimension Four

Four is the lowest dimension for which f-vectors of polytopes have not been characterized, and the same is true of flag vectors. In the late sixties and early seventies, f-vectors of 4-polytopes were studied intensively, resulting in the characterizations of the projections of f-vectors of 4-polytopes onto all pairs of components (Grünbaum [1967], Barnette-Reay [1973], Barnette [1974]). By duality only four projections need be determined.

Theorem 3.9

i. *There exists a 4-polytope P with $(f_0(P), f_1(P)) = (f_0, f_1)$ if and only if f_0 and f_1 are integers satisfying*

$$10 \leq 2f_0 \leq f_1 \leq \binom{f_0}{2}$$

and $(f_0, f_1) \notin \{(6,12),(7,14),(8,17),(10,20)\}$.

ii. *There exists a 4-polytope P with $(f_0(P), f_2(P)) = (f_0, f_2)$ if and only if f_0 and f_2 are integers satisfying*

$$1/2\left(2f_0 + 3 + (8f_0 + 9)^{1/2}\right) \leq f_2 \leq f_0^2 - 3f_0,$$

$f_2 \neq f_0^2 - 3f_0 - 1$, and $(f_0, f_2) \notin \{(6,12), (6,14), (7,13), (7,15), (8,15), (8,16), (9,16), (10,17), (11,20), (13,21)\}$.

iii. *There exists a 4-polytope P with $(f_0(P), f_3(P)) = (f_0, f_3)$ if and only if f_0 and f_3 are integers satisfying*

$$5 \leq f_0 \leq f_3(f_3 - 3)/2 \text{ and } 5 \leq f_3 \leq f_0(f_0 - 3)/2.$$

iv. *There exists a 4-polytope P with $(f_1(P), f_2(P)) = (f_1, f_2)$ if and only if f_1 and f_2 are integers satisfying*

$$f_2 \geq f_1/2 + \lceil \sqrt{f_1 + 9/4} + 1/2 \rceil + 1/2,$$

the pair (f_1, f_2) does not equal $\left(i^2 - 3i - 1, (i^2 - i)/2\right)$ for any i, and $(f_1, f_2) \notin \{(12,12), (14,13), (14,14), (15,15), (16,15), (17,15), (17,16), (18,18), (20,17), (21,19), (23,20), (24,20), (26,21)\}$.

The original proofs that the inequalities are satisfied by the f-vectors of 4-polytopes used arguments about chains of faces. The introduction of flag vectors thus simplifies the exposition of the proofs.

Here are the inequalities known to hold for the flag vectors of all 4-polytopes (Bayer [1987]).

Theorem 3.10 *Let f_0, f_1, f_2 and f_{02} be flag numbers of a 4-polytope. Then*

i. $f_{02} - 3f_2 \geq 0$.

ii. $f_{02} - 3f_1 \geq 0$.

iii. $f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0$.

iv. $6f_1 - 6f_0 - f_{02} \geq 0$.

v. $f_0 - 5 \geq 0$.

vi. $f_2 - f_1 + f_0 - 5 \geq 0$.

vii. $2(f_{02} - 3f_2) + f_1 \leq \binom{f_0}{2}$.

viii. $2(f_{02} - 3f_1) + f_2 \leq \binom{f_2 - f_1 + f_0}{2}$.

ix. $f_{02} - 4f_2 + 3f_1 - 2f_0 \leq \binom{f_0}{2}$.

x. $f_{02} + f_2 - 2f_1 - 2f_0 \leq \binom{f_2 - f_1 + f_0}{2}$.

The linear inequalities (i) and (v) are obvious; inequalities (ii) and (vi) are their duals. Inequality (iii) is Kalai's inequality (Theorem 3.8). The proofs of the other inequalities are in Bayer [1987].

The projections of the inequalities of Theorem 3.10 onto f-vectors give all the inequalities appearing in Theorem 3.9, and one other linear inequality. This inequality was conjectured by Barnette [1972].

Theorem 3.11 *If (f_0, f_1, f_2, f_3) is the f-vector of a 4-polytope, then*

$$-3f_2 + 7f_1 - 10f_0 + 10 \geq 0.$$

In Bayer [1987] the tightness of the inequalities is analyzed.

3.9 Intersection Homology

In the early parts of this section we discussed how Stanley proved the McMullen conditions using the interpretation of the h-vector as homology ranks for a toric variety. The toric variety is defined for a rational polytope even if it is not simplicial, but the singularities can be worse in this case. Different geometric realizations of the same combinatorial type of simplicial polytope have toric varieties with the same homology ranks. This is no longer the case for nonsimplicial polytopes. McConnell [1989] showed that the toric varieties associated with two different (rational) geometric realizations of the rhombododecahedron have different regular homology ranks.

The middle perversity intersection homology Betti numbers of a toric variety are, however, combinatorial invariants. A formula for these Betti numbers in terms of the face lattice of the associated polytope was given independently by Bernstein, Khovanskiĭ and MacPherson (see Stanley [1987a]). Stanley generalized these Betti number formulas to Eulerian posets. He and several other authors have applied them to the study of convex polytopes (Bayer-Klapper [1991], Kalai [1988b], Stanley [1987a]). Here the coefficients h_i are defined to agree with the original h-vector in the simplicial case; h_i represents the rank of the $(2d-2i)$th intersection homology group.

For any d-polytope P are defined a *generalized h-vector* $(h_0, h_1, \ldots, h_d) \in {}^{d+1}$, with generating function $h(P,t) = \sum_{i=0}^{d} h_i t^{d-i}$, and *$g$-vector* $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor}) \in {}^{\lfloor d/2 \rfloor + 1}$, with generating function $g(P,t) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i t^i$, related by $g_0 = h_0$ and $g_i = h_i - h_{i-1}$ for $1 \leq i \leq \lfloor d/2 \rfloor$. The generalized h-vector and g-vector are defined by the recursion

i. $g(\ , t) = h(\ , t) = 1$, and

ii. $h(P,t) = \displaystyle\sum_{\substack{G \text{ face of } P \\ G \neq P}} g(G,t)(t-1)^{d-1-\dim G}$.

We summarize the known results on generalized h-vectors in the following theorem.

Theorem 3.12

i. *The generalized h-vector of a rational polytope is the sequence of middle perversity intersection homology Betti numbers of the associated toric variety.*

ii. *The generalized h-vector of a simplicial polytope is the same as its h-vector; hence it satisfies the McMullen conditions.*

iii. *For any d-polytope, $h_0 = 1$ and for all i, $0 \leq i \leq d$, $h_i = h_{d-i}$.*

iv. *For any rational d-polytope and any i, $0 \leq i \leq d/2 - 1$, $h_i \leq h_{i+1}$.*

v. *There is a linear function from ^{e_d} to $^{d+1}$ that takes the flag vector of any d-polytope to the generalized h-vector of the polytope.*

Comments on the theorem. (i). Note that Stanley's definition of the generalized h-vector makes sense for all polytopes (or, more generally, for Eulerian posets), but the toric variety is only defined for rational polytopes. (ii). For the toric variety associated to a simplicial polytope, the middle perversity intersection homology is isomorphic to the ordinary homology. (iii). Stanley gives a purely combinatorial proof of this duality result. Thus it applies even when the toric variety is not defined. (iv). The proof of the unimodality of the generalized h-vector depends on the existence of primitive homology groups for the toric variety, hence on the rationality hypothesis. The first inequality holds trivially for all polytopes; the second is Kalai's Theorem 3.8. It is not known whether the other inequalities hold for nonrational polytopes. (iii) and (v). According to (v), (iii) gives approximately $d/2$ linear equations satisfied by the flag vectors of all d-polytopes. Theorem 3.7, however, gives all such linear equations, of which there are an exponential (in d) number. This suggests that the generalized h-vector should be embedded in a larger set of parameters equivalent to the flag vector. Kalai found one such set.

3.10 Kalai's Convolutions

Kalai [1988b] creates new parameters by applying the g-vector transformation simultaneously to different intervals of a polytope. First we give his extended definition of the g-vector as a length $d+1$ vector. Write F_d for the real vector space with basis $\{f_S : S \subseteq \{0, 1, \ldots, d-1\}\}$. An element of F_d (a "linear form") defines a real-valued function on the set of d-polytopes.

For $d \geq 0$ and $0 \leq i \leq d$, define linear functions G_i^d on the set of d-polytopes by the following recursion: $G_0^d(P) = 1$ for all P and

$$G_i^d(P) = (-1)^i \binom{d+1}{i} + \sum_{j=0}^{i-1} \sum_{s=0}^{i-j-1} (-1)^j \binom{d-i+j-s+1}{j} \sum_{\substack{F \text{ a face of } P \\ \dim F = i-j+s-1}} G_s^{i-j+s-1}(F).$$

The functions G_i^d have a natural representation as elements of F_d.

Kalai defines the *convolution* operation on the set $F = \bigcup_{d \geq 0} F_d$ as follows: for $S \subseteq \{0, 1, \ldots, d-1\}$ and $T \subseteq \{0, 1, \ldots, e-1\}$, let $f_S * f_T = f_{S \cup \{d\} \cup T + (d+1)}$, where by $T + (d+1)$ is meant $\{t + d + 1 : t \in T\}$. Thus extending linearly, the convolution of any linear forms is a linear form. Note that if P is a $(d+e+1)$-polytope, $S \subseteq \{0, 1, \ldots, d-1\}$, and $T \subseteq \{0, 1, \ldots, e-1\}$, then the convolution of f_S and f_T is

$$f_S * f_T(P) = \sum_{\substack{F \text{ face of } P \\ \dim F = d}} f_S(F) f_T(P/F).$$

Now consider the subset M_d of F_d, $M_d = \{G_{i_1}^{d_1} * G_{i_2}^{d_2} * \cdots * G_{i_k}^{d_k} : k \geq 1, 0 < i_j \leq d_j$ for $1 \leq j \leq k-1, 0 \leq i_k \leq d_k,$ and $k - 1 + \sum d_j = d\}$.

Theorem 3.13 (Kalai [1988b])

i. $|M_d| = 2^d$.

ii. M_d *is a basis for* F_d.

iii. *Every element of* M_d *defines a nonnegative function on the set of rational* d*-polytopes.*

iv. *Exactly* $2^d - e_d$ *of these functions are the zero function.*

Thus Kalai's convolutions extend the g-vector of a polytope to a vector that completely encodes the flag vector and incorporates the generalized Dehn-Sommerville equations. Note that among these are the equations $G_i^d = 0$ for $i > \lfloor d/2 \rfloor$, which are exactly the equations $h_i = h_{d-i}$.

The nonnegativity of the convolutions provides linear inequalities on the flag numbers. For any d-polytope P and its dual P^*, define $\overline{G}_i^d(P) = G_i^d(P^*)$. It is easy to see that $\overline{G}_i^d$ can be represented by an element of F_d, and that it is a nonnegative function on rational d-polytopes. Kalai suggests the following conjecture.

Conjecture 3.14 *Every linear inequality on the flag numbers of polytopes is equivalent to the nonnegativity of some nonnegative linear combination of convolutions of the* G_i^d *and the* $\overline{G}_i^d$*.*

3.11 Other Parameters

There are two other sets of parameters that extend the generalized h-vector of a d-polytope. The first is to be found in Stanley [1987a], where he first introduced the generalized h-vector. Let $T \subseteq \{0, 1, \ldots, d-1\}$, $\overline{T} = \{0, 1, \ldots, d-1\} \setminus T$, and $\mu_{\overline{T}}$ be the Möbius function of the restriction of (the face lattice of) P to elements whose dimensions are not in T. The functions ϕ_T are certain linear forms in the flag numbers.

Theorem 3.15 (Stanley [1987]) $x^{d+1}\phi_T(P, 1/x) = \phi_{\overline{T}}(P, x)$.

When $T =$ the theorem gives the equations $h_i = h_{d-i}$ on the generalized h-vector of a d-polytope. Presumably, as T ranges over all subsets of $\{0, 1, \ldots, d-1\}$, the equations of the theorem are equivalent to the generalized Dehn-Sommerville equations.

Another set of parameters comes not so directly from the h-vector. This is the *cd index* of a polytope. It was introduced by Fine, and derives from the *extended h-vector* of a polytope, mentioned in Section 3.7.

Suppose $(f_S(P))_{S \subseteq \{0,1,\ldots,n-1\}} \in {}^{2^n}$ is the flag vector of a n-polytope P. The *extended h-vector* of P is the vector $(h_S(P))_{S \subseteq \{0,1,\ldots,n-1\}} \in {}^{2^n}$ given by

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P).$$

This transformation is invertible:

$$f_S(P) = \sum_{T \subseteq S} h_T(P).$$

The extended h-vector can be given by a generating function in the algebra of polynomials in the noncommuting variables a and b. For $S \subseteq \{0, 1, \ldots, n-1\}$, write $w_i = a$ if $i \notin S$ and $w_i = b$ if $i \in S$; let $w_S = w_0 w_1 \ldots w_{n-1}$. The generating function for the extended h-vector is then $h(P) = \sum h_S(P) w_S$, the sum being over all $S \subseteq \{0, 1, \ldots, n-1\}$. Now it turns out that for every polytope P, $h(P)$ is in the subalgebra generated by $c = a + b$ and $d = ab + ba$. This fact is essentially equivalent to the generalized Dehn-Sommerville equations. The coefficients of the cd words in $h(P)$ form a vector of length e_n; this is called the *cd index* of P.

The cd index can be computed recursively via a shelling of the polytope. Like the flag vector and the extended h-vector, the cd indices of a polytope and its dual have a simple relationship: the cd index of P^* is obtained by reversing every cd word in the cd index of P. Fine conjectured that the coefficients of the cd index of any polytope are nonnegative. This was proved for quasisimplicial polytopes and their duals by Purtill [1991]. In Bayer-Klapper [1991] equations relating the cd index with the generalized h-vector are computed. They are used to give another proof of a result on the g-vectors of dual polytopes. This result was originally proved by Kalai, directly from the definition of the generalized h-vector.

Theorem 3.16 *Suppose n is even and let P and P^* be a pair of dual n-polytopes. Then $g_{n/2}(P) = g_{n/2}(P^*)$.*

3.12 Algebraic Shifting

In his characterization of f-vectors of nerves of convex sets, Kalai [1984] considered an algebraic structure analogous to the Stanley-Reisner ring of a simplicial complex and defined the notion of algebraic shifting, which we briefly describe.

Let Δ be a simplicial complex with vertices $v_1, v_2, \ldots, v_n$, and let V be an n-dimensional vector space over a field k. Choose a basis $\{e_1, e_2, \ldots, e_n\}$ of V. Form the exterior algebra $\bigwedge V$ over V. Construct the ideal I_Δ spanned by $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s} : \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\} \not\in \Delta\}$. Form the quotient algebra $\bigwedge(\Delta) = \bigwedge V/I_\Delta$. From another basis of V that is "generic" with respect to the first, another simplicial complex Δ' can be obtained that has the same f-vector as Δ but is "shifted" with respect to an appropriate partial order on subsets of $\{v_1, v_2, \ldots, v_n\}$.

Studying $\bigwedge(\Delta)$ has led to an impressive array of other results, including a simpler proof of the Upper Bound Theorem (Alon-Kalai [1985]) and a complete characterization of (f-vector, Betti sequence) pairs for simplicial complexes (Björner-Kalai [1988]).

3.13 Rigidity and Stress

Let $v_1, v_2, \ldots, v_n$ be the vertices of a d-polytope $P \subset {}^d$. An *infinitesimal motion* of the vertices is a set of vectors $u_1, u_2, \ldots, u_n$ such that $d(\|(v_i + tu_i) - (v_j + tu_j)\|^2)/dt = 0$ when $t = 0$ for all edges v_iv_j. Equivalently, $\langle (v_i - v_j), (u_i - u_j)\rangle = 0$ for all edges. Dehn [1916] proves that there are no infinitesimal motions for convex simplicial 3-polytopes apart from the rigid motions, and this extends to arbitrary $d \geq 3$ (see Whiteley [1984]).

Theorem 3.17 *For $d \geq 3$, simplicial convex d-polytopes are infinitesimally rigid.*

Given d-polytope P with vertex set V and edge set E, a *stress* is an assignment of numbers λ_{uv} to its edges $uv \in E$ such that the following equilibrium conditions hold:

$$\sum_{\{u \in V : vu \in E\}} \lambda_{vu}(v - u) = o \text{ for all } v \in V.$$

The set of stresses forms a vector space, called the *stress space.*

Dehn's Theorem is equivalent to the following.

Theorem 3.18 *Let P be a simplicial convex d-polytope, $d \geq 3$. Then the dimension of the stress space equals $f_1 - df_0 + \binom{d+1}{2}$. In particular, if $d = 3$ there are no nontrivial stresses.*

Note that the above dimension equals $h_2 - h_1$. Kalai [1987] observes that the nonnegativity of this quantity immediately yields another proof of the Lower Bound Theorem for simplicial polytopes, and he also extends the theorem to nonsimplicial polytopes (Theorem 3.8) and larger classes of complexes. See the chapter *Rigidity* by Connelly in this volume for more background on stress and rigidity in convexity.

Kalai's observation provides a new proof of the nonnegativity of $h_2 - h_1$ for simplicial d-polytopes, $d \geq 3$. On the other hand, the connection between the Stanley-Reisner ring and toric varieties proves $h_i - h_{i-1} \geq 0$ for all i, $1 \leq i \leq \lfloor d/2 \rfloor$. This foreshadowed a stronger connection between the Stanley-Reisner ring and stresses (Lee [1990, 1991a]).

Let $R_\Delta = R_1 \oplus R_2 \oplus \cdots$ be the Stanley-Reisner ring of any simplicial $(d-1)$-complex Δ with n vertices. For $\theta_1, \theta_2, \ldots, \theta_d \in R_1$ define $B = R_\Delta/(\theta_1, \theta_2, \ldots, \theta_d)$ and give B the grading induced by R_Δ. Then R_Δ is Cohen-Macaulay if and only if there exist $\theta_1, \theta_2, \ldots, \theta_d \in R_1$ such that $B = B_0 \oplus B_1 \oplus \cdots \oplus B_d$, where $h_i = \dim B_i$ for all i, $0 \leq i \leq d$. Regarding multiplication by θ_i as a linear map in R_Δ and dualizing, this condition can be reformulated.

Given $\theta_i = \sum_{j=1}^n a_{ji} x_j$ for all i, $1 \leq i \leq d$, define $v_j \in {}^d$ for all j, $1 \leq j \leq n$ by $v_j = (a_{j1}, a_{j2}, \ldots, a_{jd})$. For monomial $m = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, define $\mathrm{supp}(m) = \{x_i : r_i \neq 0\}$. For all i, $0 \leq i \leq d$, let M_i be the set of monomials of degree i. Then for all i, $1 \leq i \leq d$, a *linear i-stress* on Δ (with respect to $v_1, \ldots, v_n$) is a homogeneous polynomial $b = \sum_{m \in M_i} b_m m$ such that the following two conditions hold:

i. $b_m = 0$ if $\mathrm{supp}(m) \notin \Delta$.

ii. $\sum_{j=1}^n b_{mx_j} v_j = o$ for every $m \in M_{i-1}$.

An *affine i-stress* is defined in exactly the same way, with the additional condition that $\sum_{j=1}^n b_{mx_j} = 0$ for every $m \in M_{i-1}$. (This condition corresponds to the conjecture that $\omega = x_1 + x_2 + \cdots + x_n$ is a hyperplane section.) A linear or affine 0-stress is defined to be any real number.

Let L_i (A_i) be the vector space of all linear (affine) i-stresses. In particular, L_1 (A_1) is the set of all linear (affine) relations on the points v_j.

Theorem 3.19 (Lee [1990])

i. For simplicial $(d-1)$-complex Δ, R_Δ is Cohen-Macaulay if and only if there exist $v_1, v_2, \ldots, v_n$ such that $\dim L_i = h_i$ for all i, $0 \leq i \leq d$.

ii. Suppose Δ is a simplicial $(d-1)$-sphere. If $\dim A_i = h_i - h_{i-1}$ for all i, $1 \leq i \leq \lfloor d/2 \rfloor$, then $h = (h_0, h_1, \ldots, h_d)$ satisfies the McMullen conditions.

Condition (i) can be used for another proof that shellable simplicial complexes are Cohen-Macaulay. Using the fact that all simplicial p.l.-spheres can be obtained from the boundary of a simplex by a sequence of bistellar operations (see Section 4.3), (i) leads to a more elementary proof that simplicial p.l.-spheres are Cohen-Macaulay.

In the case that Δ is the boundary complex of a simplicial d-polytope $P \subset {}^d$ containing o in its interior, the vectors v_j can be taken to be the actual vertices of P. Then A_2 is easily seen to be isomorphic to the classical stress space, and it can be shown that (i) holds.

The polar of P sheds light on some very interesting relationships among stress, Dehn's Theorem, Minkowski's Theorem, the Brunn-Minkowski Theorem, the Hard Lefschetz Theorem, the McMullen conditions, mixed volumes, and the Alexandrov-Fenchel inequalities. See Filliman [1992] and Lee [1991a].

4 Gale Transforms and Diagrams

4.1 Introduction

Given d-polytope $P \subset {}^d$ with vertices $v_1, v_2, \ldots, v_n$, list these vectors as columns of a matrix and append a row of ones to obtain the $(d+1) \times n$ matrix A. Consider the nullspace of A, the space of all affine relations $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ on the set of vertices; i.e., $\sum_{i=1}^n \lambda_i v_i = o$ and $\sum_{i=1}^n \lambda_i = 0$. Let $\overline{A}$ be an $(n-d-1) \times n$ matrix whose rows form a basis for this space, and denote its columns by $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$. This collection $\overline{V}$ of points in ${}^{n-d-1}$ is a *Gale transform* of V. The natural correspondence between vertices v_i and transform points $\overline{v}_i$ extends to a correspondence between subsets of V and subsets of $\overline{V}$. The key property of Gale transforms is the following.

Theorem 4.1 *Let X be a proper subset of V. Then* $\mathrm{conv}(X)$ *is a face of P if and only if* $o \in \mathrm{relint}(\mathrm{conv}(\overline{V} \setminus \overline{X}))$.

A collection $\overline{W} = \{\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n\}$ is a *Gale diagram* of V if it satisfies the property given in the above theorem. For example, Gale diagrams can be obtained by scaling the points in a Gale transform independently by positive amounts.

Gale transforms and diagrams are recognized for their usefulness in establishing results when n is not much larger than d, but even in the general case they are helpful tools. Both Grünbaum [1967] and McMullen-Shephard [1971] contain good introductions. For a more extensive survey of results than is presented here, refer to McMullen [1979]. Note that the toric variety discussed in Section 3.3 is the result of an algebraic analogue of the Gale transform.

Suppose one is given $V = \{v_1, v_2, \ldots, v_n\} \subset {}^d$ such that $o \in \mathrm{int}(\mathrm{conv}(V))$, and a spherical simplicial $(d-1)$-complex Δ on these n points. The next result characterizes when Δ can be realized convexly by positively scaling the points of V.

Theorem 4.2 (Shephard [1971])
There exist positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that Δ is isomorphic to the boundary complex of $\mathrm{conv}(\{\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n\})$ *if and only if*

$$\bigcap \mathrm{relint}(\mathrm{conv}(\overline{V} \setminus \overline{F})) \neq \quad ,$$

where $\overline{V}$ is a Gale transform of V and the intersection is taken over all facets F of Δ.

A direct translation of a theorem of Bárány via Gale transforms yields the following theorem; see Bigdeli [1991].

Theorem 4.3 *Given $V = \{v_1, v_2, \ldots, v_n\} \subset \ {}^d$ in linearly general position such that $o \in \mathrm{int}(\mathrm{conv}(V))$. Let $m(V)$ be the maximum number of facets of a polytope obtained by scaling the points in V independently by positive amounts. Then $m(V) \geq \frac{1}{k^k}\binom{n}{k} + O(n^{k-1})$, where $k = n - d$.*

Another illustration of the usefulness of affine relations in the structure of polytopes is the indecomposability characterization of Smilansky [1987].

4.2 Polytopes with Few Vertices

A d-polytope P with n vertices has a Gale transform of dimension $n - d - 1$, and Sturmfels [1988] uses *affine* transforms to further reduce the dimension by one. Gale transforms have gained the (perhaps undeserved reputation) of being useful only when P has *few* vertices, i.e., when $n \leq d + 3$. In this case the Gale transform is at most two dimensional and is easier to analyze. This feeling is supported by the fact that there are many results that are easier to prove for polytopes with few vertices, and that quite often these results fail when $n = d + 4$. Here is a small sample.

Theorem 4.4 *Every $(d-1)$-dimensional p.l.-sphere with at most $d+3$ vertices is polytopal. However, there exists a simplicial 3-sphere with 8 vertices that is not polytopal.*

The first part of the theorem is due to Mani [1972] for simplicial spheres and Kleinschmidt [1976b] for nonsimplicial spheres. The nonpolytopal sphere is discussed in Grünbaum [1967] and is due to Brückner, (who, however, thought it was polytopal). Kleinschmidt [1977] proves an analogue of the above theorem for $(d-1)$-spheres with at most $2d$ vertices possessing combinatorial involutions with no fixed points.

The following theorem is stated in Grünbaum [1967].

Theorem 4.5 (Perles) *For every d-polytope with at most $d+3$ vertices there exists a combinatorially equivalent d-polytope P such that every automorphism of the boundary complex of P is induced by a geometric symmetry of P.*

Theorem 4.6 *For every d-polytope with at most $d+3$ vertices and $\epsilon > 0$ there exists a combinatorially equivalent polytope with rational vertices such that each vertex is a distance at most ϵ from the corresponding vertex of P. However, there exists a 6-polytope that is combinatorially equivalent to no polytope with rational vertices.*

The first part of the above theorem is due to Perles and stated in Grünbaum [1967], in which also a nonrational 8-polytope with 12 vertices discovered by Perles is described. Sturmfels [1987a] constructs lower dimensional examples (see Section 6.5).

One says that a facet F of a polytope P can be *preassigned* if, given any polytope F' combinatorially equivalent to F, there is a polytope P' combinatorially equivalent to P having F' as a facet corresponding to F.

Theorem 4.7 (Kleinschmidt [1976a]) *If a d-polytope has at most $d+3$ vertices, then the shape of each of its facets can be preassigned. However, there exists a 4-polytope with 8 vertices such that the shape of one of its facets cannot be preassigned.*

Gale transforms can be used to count the number of different combinatorial types of polytopes with few vertices.

Theorem 4.8 *There are $\lfloor d^2/4 \rfloor$ different combinatorial types of d-polytopes with $d+2$ vertices. There are $\lfloor d/2 \rfloor$ different combinatorial types of simplicial d-polytopes with $d+2$ vertices.*

Perles (see Grünbaum [1967]) and Lloyd [1970] count the number of simplicial and general d-polytopes with $d+3$ vertices, respectively.

4.3 Subdivisions and Triangulations

Given a d-polytope P with vertex set V, a *subdivision* of P is a collection Δ of d-polytopes such that (1) for every $Q_1, Q_2 \in \Delta$, $Q_1 \cap Q_2$ is a common face (possibly empty) of both Q_1 and Q_2; (2) P is the union of the polytopes in Δ; and (3) for every $Q \in \Delta$, the vertex set of Q is a subset of V. A subdivision Δ is a *triangulation* provided every member of Δ is a d-simplex.

The Dehn-Sommerville equations force relations among the h-vectors of a triangulation Δ, of the collection of its boundary faces $\partial\Delta$, and of the collection of its interior faces Δ^o, which also hold for general simplicial balls.

Theorem 4.9 (McMullen-Walkup [1971]) *Suppose Δ is a simplicial d-ball. Then* *i.* $h_i(\Delta) - h_{d-i+1}(\Delta) = h_i(\partial\Delta) - h_{i-1}(\partial\Delta)$ *for all* i, $1 \leq i \leq d$.

ii. $h_i(\Delta) = h_{d-i+1}(\Delta^o)$ *for all* i, $0 \leq i \leq d+1$.

A simple corollary is mentioned in Lee [1991b].

Corollary 4.10 *Suppose* Δ *is a simplicial* d*-ball. Then* $f_d(\Delta) \geq h_{\lfloor d/2 \rfloor}(\partial\Delta)$.

There is a wide range of results on subdivisions and triangulations; we mention only a few that relate to Gale transforms.

Theorem 4.11 (McMullen [1979]) *Let* P *be a* d*-polytope with vertex set* $V = \{v_1, v_2, \ldots, v_n\}$, *let* $\overline{V} \subset {}^{n-d-1}$ *be a Gale transform of* V, *and let* $\overline{z} \in {}^{n-d-1}$. *Consider the collection* Δ *of* d*-polytopes* $\mathrm{conv}(S)$ *such that* $S \subseteq V$ *and* $o \in \mathrm{relint}(\mathrm{conv}((\overline{V} \cup \{\overline{z}\}) \setminus \overline{S}))$. *Then* Δ *is a subdivision of* P.

Subdivisions and triangulations of the above form are called *regular*. An equivalent way to generate regular subdivisions of a d-polytope $P \subset {}^d$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ is to choose real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and form the convex hull Q of $\{(v_1, \lambda_1), (v_2, \lambda_2), \ldots, (v_n, \lambda_n)\}$. Projecting the facets in the upper hull of Q into $({}^d, 0)$ yields a regular subdivision of P.

We need some definitions. The *link* of a face F in a simplicial complex Δ is the set of faces G of Δ such that $F \cap G =$ and $F \cup G$ is a face of Δ. The *stellar subdivision* of F in Δ is obtained by removing F from Δ and adding a new vertex v along with all simplices formed from v, a proper subface of F, and a face in the link of F. A *bistellar* operation on a simplicial sphere is a certain combination of a stellar subdivision and inverse stellar subdivision at the same site. The following is then a consequence of a line shelling of the upper hull of Q.

Theorem 4.12 (Ewald [1978]) *The boundary complex of any simplicial* d*-polytope can be obtained from that of a* d*-simplex by a sequence of bistellar operations, such that at each intermediate stage the simplicial complex is polytopal.*

Pachner [1990] proves that a simplicial complex is a p.l.-sphere if and only if it is obtainable from the boundary complex of a simplex by a sequence of bistellar operations. In fact, Pachner shows that simplicial p.l.-spheres are precisely boundaries of shellable balls. However, the undecidability result mentioned in Section 2.10 implies that for simplicial p.l.-spheres, unlike for polytopes, no upper bound on the number of such operations needed can be computed from the given simplicial complex. On the other hand, properties of the h-vector imply that if Σ is a $(d-1)$-dimensional simplicial p.l.-sphere, then at least $h_{\lfloor d/2 \rfloor}(\Sigma)$ bistellar operations are necessary, generalizing Corollary 4.10.

Let P be a d-polytope, F be a facet of P with supporting hyperplane H, and x be a point in ^{d}. Then x is *beyond* F if x and the interior of P lie in opposite open halfspaces determined by H, and *beneath* F if x and the interior of P lie in the same open halfspace determined by H. Now suppose that x lies beyond precisely one facet F of P and beneath all the others. Let $\mathcal{C}$ denote the boundary complex of P excluding F. Projecting $\mathcal{C}$ onto H centrally through x results in a polyhedral $(d-1)$-complex isomorphic to $\mathcal{C}$, called a *Schlegel diagram* of P. See Grünbaum [1967]. Gale transforms provide a characterization of Schlegel diagrams.

Theorem 4.13 (Sturmfels [1986]) *Let Δ be a subdivision of a convex $(d-1)$-polytope P, this time allowing the vertex set V of Δ to be a strict superset of the vertex set W of P, but on the other hand requiring that proper faces of P be faces of Δ. Then Δ is the Schlegel diagram of some d-polytope if and only if*

$$(\bigcap \mathrm{relint}(\mathrm{pos}(\overline{V} \setminus \overline{F}))) \ \cap \ (-\mathrm{relint}(\mathrm{pos}(\overline{V} \setminus \overline{W}))) \neq \quad ,$$

where $\overline{V}$ is a Gale transform of V and the first intersection is taken over all $(d-1)$-polytopes F of Δ.

Not all subdivisions Δ of a $(d-1)$-polytope P satisfying the conditions in the first sentence of the theorem are Schlegel diagrams, even when P is two dimensional. However, if P is two dimensional, then Δ is isomorphic to the Schlegel diagram of some 3-polytope (Grünbaum [1967]).

For polytopes with few vertices all subdivisions are regular, but this is not true in general.

Theorem 4.14 (Lee [1991b]) *If P is a d-polytope with at most $d+3$ vertices, then every subdivision of P is regular. However, there exist 3-polytopes with 7 vertices that possess nonregular triangulations.*

On the other hand, the collection of all regular subdivisions of a given polytope has a nice structure, discovered by Gel'fand, Kapranov, and Zelevinskiĭ in connection with their work on generalized discriminants and determinants.

Theorem 4.15 (Gel'fand, Kapranov, Zelevinskiĭ [1989]) *The collection of all regular subdivisions of a given d-polytope P with n vertices, partially ordered by refinement, is combinatorially equivalent to the boundary complex of some $(n-d-1)$-polytope Q.*

This polytope Q is called the *secondary polytope* of P, and can be constructed as follows. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of P, and for any triangulation Δ of P (whether regular or not) define $z(\Delta) = (z_1, z_2, \ldots, z_n) \in {}^n$ by $z_i = \sum \mathrm{vol}(F)$, where the sum is taken over all d-simplices F containing v_i. Then $Q = \mathrm{conv}(\{z(\Delta) : \Delta \text{ is a triangulation of } P\})$. In particular, the vertices of Q correspond to the regular triangulations of P. Alternate constructions are described in Billera-Filliman-Sturmfels [1990]. For example, Q can be expressed as a discrete or continuous Minkowski sum of polars of polytopes corresponding to various translates of a Gale transform of P. Generalizations of the secondary polytope appear in Billera-Sturmfels [1990].

In Section 3.4 we learned that the h-vector of a simplicial d-polytope P satisfies $h_k \geq h_{k-1}$ for all k, $1 \leq k \leq \lfloor d/2 \rfloor$. From the Lower Bound Theorem (2.11) we also know that $h_2 = h_1$ if and only if P is stacked. In general, P is called *k-stacked* if P has a triangulation such that there is no interior face of dimension less than $d-k$. The McMullen conditions and Theorem 4.9 imply that if P is k-stacked, then $h_k = h_{k-1}$. McMullen and Walkup conjectured the converse as part of their Generalized Lower-Bound Conjecture.

Conjecture 4.16 (McMullen-Walkup [1971]) *Let P be a simplicial d-polytope. For all k, $1 \leq k \leq \lfloor d/2 \rfloor$, $h_k = h_{k-1}$ if and only if P is $(k-1)$-stacked.*

The following is a consequence of the construction in Billera-Lee [1981b].

Theorem 4.17 (Kleinschmidt-Lee [1984]) *Let P be a simplicial d-polytope such that $h_k = h_{k-1}$ for some k, $1 \leq k \leq \lfloor d/2 \rfloor$. Then there exists a k-stacked simplicial d-polytope Q with the same h-vector as P.*

A few cases of the conjecture have been resolved by interpreting the differences $h_k - h_{k-1}$ as winding numbers in Gale transforms, but as a whole the conjecture remains unresolved.

Theorem 4.18 (Lee [1991c]) *The above conjecture holds if $f_0 \leq d+3$ or if $k < f_0/(f_0-d)$.*

Suppose P is a d-polytope. Some particular regular triangulations of P, called *pulling* triangulations, can be obtained by first ordering the vertex set of P, $V = \{v_1, v_2, \ldots, v_n\}$. For every j-face F of P let $v(F)$ denote the vertex of smallest index that is in F. A *full flag* of P is a chain of faces $F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_d = P$ such that $\dim F_j = j$ for all j, $0 \leq j \leq d$, and $v(F_j) \notin F_{j-1}$ for all j, $1 \leq j \leq d$. Associate with each full flag the simplex $\mathrm{conv}(\{v(F_0), v(F_1), \ldots, v(F_d)\})$. Then these simplices are all d-dimensional and together determine a triangulation of P. This idea appears in Hudson [1969] in a more general context and has been frequently rediscovered in various guises.

Write $J(P,t) = 1 + \sum_{n=1}^{\infty} i(P,n)t^n$, where $i(P,n)$ denotes the number of points $x \in P$ such that $nx \in \ ^d$. Stanley uses pulling triangulations to prove the following, which strengthens earlier work of Ehrhart and McMullen.

Theorem 4.19 (Stanley [1980a]) *Suppose every vertex of P is integral. Then $J(P,t) = W(P,t)/(1-t)^{d+1}$ where $W(P,t)$ is a polynomial of degree at most d with nonnegative integer coefficients.*

For integral d-polytope P, call an ordering σ of its vertices *compressed* if every d-simplex in the associated pulling triangulation has volume $1/d!$. P itself is *compressed* if every ordering σ is compressed. For example, the unit d-cube is compressed.

Theorem 4.20 (Stanley [1980a]) *If P is an integral d-polytope with compressed ordering σ, then*

$$i(P,n) = \sum_{i=0}^{d} \binom{n-1}{i} f_i(\Delta),$$

and $W(P,t) = h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d$, where Δ is the pulling triangulation induced by σ.

(Compare the above with Equation (5) of Section 3.2.)

Corollary 4.21 (Stanley [1980a]) *If P is a compressed integral d-polytope and σ is an ordering, then the f-vector of the triangulation induced by σ depends only on P, not on σ.*

The Cartesian product $T^m \times T^n$ of two simplices of any dimension is compressed, but unlike the d-cube has the property that *all* of its triangulations have the same f-vector, whether induced by an ordering as above or not; see Billera-Cushman-Sanders [1988]. Any polytope with this property is called *equidecomposable.* A *weakly neighborly* polytope is one for which every set of $k+1$ vertices is contained in a face of dimension at most $2k$. $T^m \times T^n$ is also weakly neighborly. The following results are drawn from Bayer [1990] and Stanley [1991].

Theorem 4.22

i. If P is a rational d-polytope and Δ is any subdivision of P, then $h(\Delta) \geq h(P)$, where h is the generalized h-vector, and P is regarded as a d-complex.

ii. If P is a weakly neighborly d-polytope, then P is equidecomposable, and $h(\Delta) = h(P)$ for any triangulation Δ of P.

iii. If P is a rational weakly neighborly d-polytope and Δ is any subdivision of P, then $h(\Delta) = h(P)$.

iv. If P is a rational d-polytope and $h(\Delta) = h(P)$ for all triangulations of P, then P is weakly neighborly.

Bayer [1990] uses Gale transforms to characterize equidecomposable and weakly neighborly d-polytopes with at most $d+3$ vertices. All 2-polytopes are weakly neighborly. A 3-polytope is weakly neighborly iff it is a prism over a triangle or a pyramid over a polygon. A simplicial polytope is weakly neighborly iff it is a simplex or an even dimensional neighborly polytope. Other classes of weakly neighborly polytopes include pyramids over weakly neighborly polytopes, subpolytopes of weakly neighborly polytopes, and Lawrence polytopes (Section 4.5).

4.4 Oriented Matroids

Matroids and oriented matroids provide a setting for a combinatorial abstraction of convexity, including analogues of Carathéodory's theorem, Radon's theorem, Helly's theorem, and the Hahn-Banach theorem; generalizations of point and hyperplane arrangements, convex polytopes, and Gale transforms; as well as a combinatorial derivation of linear programming. See the chapter *Oriented Matroids* by Bokowski in this volume or Björner-Las Vergnas-Sturmfels-White-Ziegler [1991] for details. Sturmfels [1986] discusses the relationship between oriented matroids and Gale transforms.

A *matroid* M is a pair $(E, \mathcal{C})$ consisting of a finite set E and a collection of nonempty incomparable subsets $\mathcal{C}$ of E (called the *circuits* of M) satisfying the following property: $C_1, C_2 \in \mathcal{C}$, $e \in C_1 \cap C_2$, and $e' \in C_1 \setminus C_2$ implies the existence of $C_3 \in \mathcal{C}$ such that $e' \in C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$. For example, the collection of supports of elementary vectors in a subspace V of $\ ^n$ forms the circuits of a matroid on $E = \{1, 2, \ldots, n\}$. Given matroid (E, C), let $\mathcal{C}^*$ be the collection of all minimal nonempty subsets C^* of E such that $|C^* \cap C| \neq 1$ for all $C \in \mathcal{C}$. Then $M^* = (E, \mathcal{C}^*)$ is also a matroid, called the *dual* of M, and members of $\mathcal{C}^*$ are called the *cocircuits* of M. In the preceding example, $\mathcal{C}^*$ is the collection of supports of elementary vectors in $V^\perp$. When a matroid can be derived from a subspace of $\ ^n$, it is called *representable* (over). So matroids provide a generalization of unsigned patterns of dependences of finite collections of vectors.

Oriented matroids, on the other hand generalize *signed* patterns of dependences. Let E be a finite set. A *signed set* X is an ordered pair (X^+, X^-) of disjoint subsets of E. The set $\underline{X} = X^+ \cup X^-$ is called the *underlying set* of X, and by $-X$ is meant (X^-, X^+). Two signed sets X, Y are said to be *orthogonal* if either their underlying sets are disjoint or else both $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq$. Let $\mathcal{O}, \mathcal{O}^*$ be two collections of signed sets in E. Then $M = (E, \mathcal{O})$ and $M^* = (E, \mathcal{O}^*)$ is a *dual pair of oriented matroids* provided the following conditions hold:

i. The underlying sets of the members of $\mathcal{O}$ (respectively, $\mathcal{O}^*$) form the circuits (respectively, cocircuits) of a matroid (called the *underlying matroid* $\underline{M}$).

ii. $X \in \mathcal{O}$ (respectively, $\mathcal{O}^*$) implies $-X \in \mathcal{O}$ (respectively, $\mathcal{O}^*$).

iii. If $X, Y \in \mathcal{O}$ (respectively, $\mathcal{O}^*$) and $\underline{X} = \underline{Y}$, then $Y = \pm X$.

iv. If $X \in \mathcal{O}$ and $Y \in \mathcal{O}^*$, then X and Y are orthogonal.

As in the unoriented case, members of $\mathcal{O}$ are referred to as the *circuits* of M and members of $\mathcal{O}^*$ as the *cocircuits*.

So, for example, the signed supports of elementary vectors in a pair $V, V^\perp$ of dual subspaces of $\ ^n$ form the circuits and cocircuits of an oriented matroid on $\{1, 2, \ldots, n\}$. This is true in particular for the null spaces of A and $\overline{A}$ used in Section 4.1 to define the Gale transform of a polytope, and suggests the following definition.

Let $\mathcal{O}^*$ be the set of cocircuits of an oriented matroid M on E. A cocircuit X is *positive* if $X^- =\ $, and M is *acyclic* if it has positive cocircuits. The *facets* of M are the sets of the form $E \setminus \underline{C}$ where C is a positive cocircuit of E. The *faces* of M are the intersections of finite numbers of facets of M. The collection of faces of M, ordered by inclusion, forms a lattice, called the *(Las Vergnas) face lattice* of M. Vertices of the lattice are faces of M that have rank one in the underlying matroid $\underline{M}$, and M is a *matroid polytope* provided all one-element subsets of E are vertices. In the case that M is derived from a dual pair of subspaces of $\ ^n$, M is called *representable* (over) and the face lattice of M is isomorphic to the face lattice of a convex polytope.

Oriented matroids can alternately be defined by assigning sign patterns to ordered bases of a matroid (maximal subsets of E containing no circuit) which would not be inconsistent with the Plücker-Graßman relations should the oriented matroid be representable. All bases have the same cardinality, called the *rank* of the matroid. Much of the usefulness of oriented matroids in the theory of convex polytopes is related to realizability results, some of which are also discussed in Section 6.5. Bokowski and Sturmfels [1987] developed algorithms based upon oriented matroids to test polytopality of spheres, which, combined with other results, has led to a complete classification of simplicial neighborly 3-spheres with 10 vertices into polytopal and nonpolytopal spheres.

We mention a few other results that are obtainable by matroid techniques. All of them are quoted from Björner-Las Vergnas-Sturmfels-White-Ziegler [1991].

Theorem 4.23 (Las Vergnas [1986]) *For $d \geq 2$ there exists a set of $(d+1)(d+2)/2$ points in general position in $\ ^d$ which is not projectively equivalent to the set of vertices of any d-polytope.*

The next theorem can be found in Cordovil-Duchet [1987].

Theorem 4.24 (Duchet-Roudneff) *Let n, d be integers with $n \geq d+1 \geq 3$. There exists an integer $N = N(n,d)$ such that every set of N points in general position in affine d-space contains the n vertices of a cyclic d-polytope. Moreover, cyclic polytopes are the only combinatorial types of polytopes with this property.*

Theorem 4.25 (Sturmfels [1987b]) *Suppose the convex hull of $\{v_1, v_2, \ldots, v_n\}$ in $\ ^d$ is combinatorially equivalent to $C(n,d)$.*

Then there exists a curve C containing $v_1, v_2, \ldots, v_n$ such that every hyperplane in ^{d} meets C in at most d points.

Theorem 4.26 (Grünbaum [1967])

i. *Let M be a neighborly rank $2k+1$ matroid polytope with $n \leq 2k+3$. Then M is isomorphic to $C(n, 2k+1)$.*

ii. *For all $k \geq 2$ there is a representable neighborly rank $2k+1$ matroid polytope with $2k+4$ vertices which is not isomorphic to a cyclic polytope.*

A matroid polytope M is called *rigid* if M is determined by its face lattice.

Theorem 4.27 (Shemer [1982]) *Every neighborly rank $2k+1$ matroid polytope is rigid.*

Theorem 4.28 (Bokowski-Sturmfels [1987]) *There exists a triangulated 3-sphere (the Barnette sphere) with 8 vertices which is not the face lattice of any matroid polytope.*

Theorem 4.29 *For all $d \geq 4$ and $n \geq d+5$ there exists a rank $d+1$ matroid polytope whose face lattice is a nonpolytopal simplicial $(d-1)$-sphere with n vertices.*

4.5 Lawrence Polytopes

Bayer-Sturmfels [1990] is a good reference for the Lawrence construction, which provides an effective method for lifting matroid representability results into polytope realizability results. See also Björner-Las Vergnas-Sturmfels-White-Ziegler [1991]. A polytope is called a *Lawrence polytope* if it has a centrally symmetric Gale transform. Let $\{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n\}$ be a Gale transform of convex d-polytope P with n vertices. Let $\Lambda(P)$ be a polytope whose Gale transform is $\{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n, -\overline{v}_1, -\overline{v}_2, \ldots, -\overline{v}_n\}$. Then $\Lambda(P)$ is a Lawrence $(d+n)$-polytope with $2n$ vertices which contains P as a quotient polytope. Hence every polytope is the quotient of a Lawrence polytope.

This construction can be extended in a natural way to oriented matroids M, so that if M is a rank r oriented matroid on n elements, then $\Lambda(M)$ is a rank $n+r$ oriented matroid on $2n$ elements. It turns out that the combinatorial structure of the face lattice of $\Lambda(M)$ depends strongly upon the matroid structure of M.

Theorem 4.30 (Bokowski-Sturmfels [1987]) *Every Lawrence matroid polytope $\Lambda(M)$ is rigid. I.e., $\Lambda(M)$ is uniquely determined by its face lattice.*

Theorem 4.31 (Bayer-Sturmfels [1990]) *For any oriented matroid M, the f-vector and the flag vector of the Lawrence polytope $\Lambda(M)$ are functions of the underlying matroid $\underline{M}$. In the generic case where M is a uniform oriented matroid of rank r on n points, the f-vector and the flag vector can be expressed as functions depending only on n and r.*

Theorem 4.32 (Billera-Munson [1984])

i. The face lattice of $\Lambda(M)$ is polytopal if and only if M is representable.

ii. There exists a rank 12 matroid polytope with 16 vertices whose face lattice is not polytopal.

iii. There exists a rank 12 matroid polytope M with 16 vertices which does not have a polar; i.e., there is no matroid polytope having a face lattice anti-isomorphic to the face lattice of M.

Although oriented matroids capture and generalize the combinatorial flavor of convex polytopes very well, it is curious that the existence of polars does not generalize. Other realization problems will be mentioned in Section 6.5.

Theorem 4.33 (Bayer-Sturmfels [1990])

i. The convex realization space of the face lattice of $\Lambda(M)$ is homotopy equivalent to the realization space of M.

ii. There exist two combinatorially equivalent Lawrence 19-polytopes P and Q that are not isotopy equivalent.

We will return to the isotopy problem in Section 6.4.

5 Graphs of Polytopes

5.1 Introduction

In this section we mention only briefly the major results on the graphs of polytopes. The general subject is covered in Grünbaum [1975], and Klee-Kleinschmidt [1987] is an extensive survey of the d-step conjecture. See also Klee-Kleinschmidt [1991].

5.2 Steinitz's Theorem

The *graph* of a polytope is the set of vertices and edges of the polytope. The earliest major result on the graphs of polytopes was Steinitz's Theorem on the graphs of 3-polytopes. Recall that a graph is *d-connected* if every pair of vertices is connected by d internally disjoint paths or, equivalently, the removal of any

$d-1$ vertices leaves a connected graph with at least two vertices. A graph is *planar* if it can be represented in ^{2} by a set of distinct points (for the vertices) and a set of curves (for the edges) which intersect only at common vertices.

Theorem 5.1 (Steinitz [1922]) *A graph G is the graph of a 3-polytope P if and only if it is planar and 3-connected.*

See also Steinitz-Rademacher [1934] and Grünbaum [1967].

Mani [1971] showed that P can be chosen so that the isometries of P correspond to the automorphisms of the graph. Steinitz's Theorem has many other consequences on the realizability of 2-dimensional complexes (see Section 6.5).

The connectivity condition can be extended to polytopes of arbitrary dimension.

Theorem 5.2 (Balinski [1961]) *The graph of every d-polytope is d-connected.*

Another connectivity criterion is due to Klee (see Grünbaum [1967]). For a polytope P with at least $n+1$ vertices, define $s_n(P)$ to be the maximum number of connected components that can remain when n vertices are removed from the graph of P, and $s(n,d) = \max\{s_n(P) : P \text{ is a } d\text{-polytope}\}$.

Theorem 5.3 (Klee [1964]) *For all n and d,*

$$s(n,d) = \begin{cases} 1, & \text{if } n \leq d-1, \\ 2, & \text{if } n = d, \\ f_{d-1}(C(n,d)), & \text{if } n \geq d+1. \end{cases}$$

Klee then used the above result to show that for every d there is a graph of a d-polytope that cannot be the graph of an e-polytope for any $e \neq d$. Moreover, such *dimensionally unambiguous* graphs can have arbitrarily large numbers of vertices.

The *k-skeleton* of a d-polytope is the polyhedral complex generated by the k-faces of the polytope. Thus the 1-skeleton is the graph of the polytope. The following result, originally observed for graphs, was proved in general by Grünbaum.

Theorem 5.4 (Grünbaum [1965]) *The k-skeleton of a d-polytope $(1 \leq k \leq d-1)$ contains a subdivision of the k-skeleton of the d-simplex.*

When does the graph of a polytope determine the entire combinatorial structure of the polytope? Steinitz's Theorem implies that it does when the polytope is 3-dimensional. In general this is not the case, however. For example all neighborly polytopes with the same number of vertices have the same graph, namely, the complete graph. (For neighborly polytopes, see Grünbaum [1967].) Recall that Theorem 2.9 of Section 2.6 shows that the graph of a *simple* polytope uniquely determines its face lattice. The same is true for zonotopes (Minkowski sums of finite collections of line segments) (Björner-Edelman-Ziegler [1990]). These cases are quite special. In general, one cannot reconstruct a d-polytope even from its $(d-3)$-skeleton. The $(d-2)$-skeleton does, however, determine the combinatorial type of any d-polytope, and the same is true for the $\lfloor d/2 \rfloor$-skeleton of a simplicial d-polytope. See Grünbaum [1967] and Perles [1970].

5.3 Hamiltonian Circuits

An important issue in graph theory is the existence of Hamiltonian circuits (closed paths containing all vertices), which began with Hamilton's observations about circuits on the dodecahedron. It is natural, therefore, to ask whether graphs of polytopes have Hamiltonian circuits. Already in the last century Kirkman knew of polytopes without Hamiltonian circuits. Tutte [1946] found the first example of a simple polytope without a Hamiltonian circuit. His example is a 3-polytope; it is still open whether all simple polytopes of dimension higher than three have Hamiltonian circuits. The following classes of 3-polytopes are known, however, to have Hamiltonian circuits: those with 4-connected graphs; simple 3-polytopes with at most 36 vertices; simple 3-polytopes with at most two types of 2-faces: 3-gons, 4-gons or 6-gons; and simplicial 3-polytopes with maximum vertex degree six. See Grünbaum [1967] and Klee-Kleinschmidt [1991]. A Hamiltonian path in a graph is a spanning tree with maximum degree two. Thus, the following theorem is related in a natural way.

Theorem 5.5 (Barnette [1966]) *The graph of every 3-polytope has a spanning tree with maximum degree 3.*

Given integers n, d such that $n > d \geq 3$, is there any simple d-polytope with n vertices that has a Hamiltonian circuit? A more basic question is: Does there even exist a simple d-polytope with n vertices? Of course, the relation $f_1 = dn/2$ shows that n must be even if d is odd. The McMullen conditions provide other restrictions on the possible values of n. Perles and Prabhu (see Prabhu [1991]) address both questions simultaneously.

Theorem 5.6 (Perles-Prabhu)

- *i. There is a constant c such that for all $n > cd^{5/2}$ (where n is even if d is odd) there exists a simple d-polytope with n vertices that has a Hamiltonian circuit.*
- *ii. For all d there exists an integer $n(d) = O(d^{5/2})$ (where $n(d)$ is even if d is odd) such that there is no simple d-polytope with $n(d)$ vertices.*

5.4 Diameter

Probably the most intensively studied question on polytope graphs is that of the diameter (see Klee-Kleinschmidt [1987]). The *diameter* $\delta(P)$ of a polytope

P is the maximum length of a shortest edge-path between two vertices of the polytope. Write $\Delta(d,n)$ for the maximum diameter of d-polytopes with n facets.

Much of the interest in diameters of polytopes comes from the search for efficient linear programming algorithms. If the function $\Delta(d,n)$ is not bounded by a polynomial in d and n, then no edge-following linear programming algorithm with arbitrary starting vertex could have polynomial complexity. A proof of a polynomial bound for $\Delta(d,n)$ might, on the other hand, suggest an efficient linear programming algorithm.

In computing Δ we can restrict the class of polytopes.

Theorem 5.7

i. *For $n > d \geq 2$, $\Delta(d,n)$ is the maximum diameter of a simple d-polytope with n facets.*

ii. *For $n \geq 2d \geq 4$, $\Delta(d,n)$ is realized as the distance between two vertices not on a common facet, in a simple d-polytope with n facets.*

Following are equivalent conjectures concerning the diameter.

Conjecture 5.8

i. Hirsch Conjecture (Dantzig [1963]). *For $n > d \geq 2$, $\Delta(d,n) \leq n-d$.*

ii. d-Step Conjecture (Dantzig [1963]). *For $d \geq 2$, $\Delta(d,2d) = d$.*

iii. Nonrevisiting Conjecture (Klee-Walkup [1967]). *Between any two vertices of a simple polytope, there is a path that does not revisit any facet.*

Remarkably little is known about these conjectures. They can fail for spheres (Walkup [1978]) and for unbounded pointed polyhedra (Klee-Walkup [1967]), though no d-polyhedron with n facets is known with diameter greater than $2n-2d$. For many values of d and n there are d-polytopes with n facets with diameter equal $n-d$ (e.g., d-cubes). Barnette [1969] found an upper bound for Δ for general d and n that is exponential in d, and then Larman [1970] obtained a better (but still exponential) bound. This was recently improved by Kalai [1990c] and further by Kalai-Kleitman [1992]. The lower bound below is due to Adler [1974].

Theorem 5.9 *For $n > d \geq 2$,*

i. $\Delta(d,n) \leq \min\{n2^{d-3}, n^{2+\log d}\}$.

ii. $\Delta(d,n) \geq \lfloor (n-d) - \frac{n-d}{\lfloor 5d/4 \rfloor} \rfloor - 1$.

Precise values of $\Delta(d,n)$ are known only for small d and n (see Klee-Kleinschmidt [1987]).

Theorem 5.10

i. $\Delta(d,n) = \lfloor (d-1)n/d \rfloor - d + 2$, *if* $d \leq 3$ *or* $n \leq d+4$.

ii. $\Delta(4,9) = \Delta(4,10) = 5$, $\Delta(5,9) = 4$, *and* $\Delta(5,11) = 6$.

Note that when $n-d \leq d \leq 5$, $\Delta(d,n)$ attains the Hirsch bound, $\Delta(d,n) = n-d$.

The final result of this section is an easy consequence of a particular class of regular triangulations.

Theorem 5.11 (Lee [1991b]) *Every simple d-polytope P with n facets can be realized as a facet of a simple $(d+1)$-polytope Q with $n+1$ facets such that the diameter of Q does not exceed $2n-2d$.*

The proof of the theorem implies that, given any linear program with bounded feasible region and arbitrary starting vertex, by augmenting the problem with one variable and one constraint an optimum vertex can be reached in a linear number of pivots.

6 Combinatorial Structure

6.1 Introduction

Finally we come to the broadest problem, that of classifying the combinatorial types of all polytopes. This section deals with asymptotic formulas for the number of combinatorial types, isotopy, and the realization of types of spheres as polytopes, rational polytopes and spherical polytopes. We discuss equifacetted polytopes, barycentric subdivisions, and the numbers of n-gons in a 3-polytope. We cover only briefly some topics discussed in greater depth in Klee-Kleinschmidt [1991].

6.2 Regular Polytopes

There has been some speculation that the regular or Platonic solids were the primary motivation for the *Elements* of Euclid. The symmetry group of P is *flag transitive* if, for any two flags of P, there exists a symmetry which maps one flag onto the other. The polytope P is *regular* if its symmetry group is flag transitive. It is *semiregular* if it is not regular, but each of its facets is regular and the symmetry group of P is vertex transitive. See Coxeter [1963].

In three dimensions, Euler's relation easily implies that the only regular polytopes are the five Platonic solids. The three dimensional semiregular polytopes consist of the thirteen Archimedean solids, together with the two infinite classes of the prisms and the antiprisms.

Theorem 6.1 *Up to rigid motion and scaling, there are five regular 3-polytopes and six regular 4-polytopes. For all dimensions $d > 4$ there are only three regular d-polytopes: the d-cube, the d-crosspolytope, and the regular d-simplex.*

6.3 Numbers of Combinatorial Types

See Klee-Kleinschmidt [1991] for a good summary of this topic. We briefly mention what is found there.

The combinatorial types of 3-polytopes are well-understood. By Steinitz's Theorem, classifying 3-polytopes is equivalent to classifying 3-connected planar graphs. Exact numbers of combinatorial types of 3-polytopes with at most 22 edges are given in Duivestijn and Federico [1981]. Asymptotic formulas in terms of number of edges, number of vertices or numbers of facets and vertices are summarized in Bender [1987]; these are the product of several people's work over 25 years.

Theorem 6.2 *The number of combinatorial types of* 3*-polytopes with* $i+1$ *vertices and* $j+1$ *facets is asymptotically*

$$\frac{1}{972ij(i+j)}\binom{2i}{j+3}\binom{2j}{i+3}.$$

Gale diagrams have been used to count the combinatorial types of simplicial or arbitrary polytopes with $d+2$ or $d+3$ vertices; see Section 4.2. A d-polytope is *neighborly* if every set of $\lfloor d/2 \rfloor$ vertices forms a face. For even dimension d, every neighborly d-polytope with $d+2$ or $d+3$ vertices is equivalent to a cyclic polytope. The face lattice of a cyclic polytope is specified by Gale's evenness criterion (see Grünbaum [1967]). For odd d, the numbers of types of neighborly d-polytopes with at most $d+3$ vertices are given in McMullen [1974]. The numbers of d-polytopes and neighborly d-polytopes with $d+4$ vertices are known only for $d \leq 4$.

Asymptotic upper and lower bounds for the number of combinatorial types of d-polytopes have been brought surprisingly close in the last few years. Let $c(n,d)$ be the number of types of d-polytopes with n vertices, $c_s(n,d)$ the number of these that are simplicial.

Theorem 6.3

i. $\left(\frac{n-d}{d}\right)^{nd/4} \leq c_s(n,d) \leq c(n,d) \leq (n/d)^{d^2 n(1+O(\frac{1}{\log(n/d)}+\frac{\log\log(n/d)}{d\log(n/d)}))}$.

ii. $c(n,d) \leq 2^{n^3+O(n^2)}$.

This is based on Shemer's [1982] estimate of the number of simplicial neighborly polytopes, and Goodman's and Pollack's [1986] application to configurations of Betti number estimates by Milnor, with improvements by Alon [1986]. Comparing this with Theorem 2.14, we see that there are many more spheres than polytopes.

6.4 Isotopy

Björner-Las Vergnas-Sturmfels-White-Ziegler [1991] is a good reference for the isotopy problem. Steinitz (Steinitz-Rademacher [1934]) proved the isotopy property for 3-polytopes. Represent a polytope in ^{3} with n vertices in some order, $v_1, v_2, \ldots, v_n$ by a length $3n$ vector $(v_1, v_2, \ldots, v_n)$.

Theorem 6.4 *Suppose P and Q are two combinatorially equivalent polytopes in ^{3} with n vertices in corresponding order. Then there is a path in ^{3n} connecting P with either Q or the reflection of Q, such that each point of the path represents a polytope combinatorially equivalent to P.*

This isotopy property fails in dimensions higher than 3 (see Section 4.5). Already in ^{4} there is a simplicial polytope with 10 vertices for which it fails. The polytope was first described in Bokowski-Ewald-Kleinschmidt [1984]; that it fails the isotopy property is due to Mnëv [1988] and Bokowski and Guedes de Oliveira [1990]; for a good account see Bokowski-Sturmfels [1989].

A group of mathematicians in Leningrad (Viro [1988]) has worked on a more general study of realization spaces. The *realization space* of a combinatorial type of polytope is the set of vector representations of all realizations of the combinatorial type. A polytope satisfies the isotopy property if its realization space is connected. Mnëv showed that the general situation is very far from the isotopy property.

Theorem 6.5 (Mnëv [1988]) *For any semi-algebraic variety V there exists a convex polytope whose realization space is homotopy equivalent to V.*

6.5 Realization

Steinitz's Theorem says that every polyhedral complex homeomorphic to the 2-dimensional sphere can be realized convexly, i.e., as the boundary of a convex 3-polytope. In Section 4.2 we saw that the same was true for $(d-1)$-dimensional p.l.-spheres with at most $d+3$ vertices. There are two simplicial and forty nonsimplicial 3-spheres with 8 vertices that cannot be realized as boundaries of polytopes; see Altshuler-Steinberg [1985] for a complete list of these. Klee and Kleinschmidt [1991] summarize the numbers of polytopal and nonpolytopal spheres of various types.

Tarski's decision method (Grünbaum [1967]) implies the following.

Theorem 6.6 *There is a decision procedure to determine whether a given complex is polytopal or not.*

However, this method is far from efficient. Though the techniques of oriented matroids (Section 4.4) are much better, still they cannot handle spheres with large numbers of vertices relative to the dimension. Further, Sturmfels proved that the polytopality of a sphere (of dimension at least 5) cannot be determined locally.

Theorem 6.7 (Sturmfels [1987a]) *For infinitely many different nonpolytopal 5-spheres, every subcomplex on fewer vertices can be extended to the boundary of a polytope.*

In Section 4.2 it was noted that all d-polytopes with at most $d+3$ vertices can be realized rationally, i.e., with vertices in ^{d}. This is also the case for all simplicial polytopes, and also for all 4-polytopes with at most 8 vertices (Altshuler-Steinberg [1985]). Another consequence of Steinitz's Theorem is that the same holds for 3-polytopes.

Theorem 6.8 *All combinatorial types of 3-polytopes can be realized with rational vertices.*

We do not know if this continues to hold for 4- and 5-polytopes, but it fails in higher dimensions.

Theorem 6.9 (Sturmfels [1987c]) *The decidability of the existence of a rational realization of a lattice as the face lattice of a polytope is equivalent to the decidability of the existence of rational roots of polynomials with integer coefficients.*

Theorem 6.10

i. All d-polytopes can be realized in ^{d}, where is the field of real algebraic numbers.

ii. For every proper subfield Φ of , there is a 6-polytope not realizable in Φ^6.

Part (i) of the above theorem is due to Lindström [1971], and part (ii) to Sturmfels [1987a].

There are two main questions concerning the realization of facets of polytopes. One asks whether the shape of a facet F of a d-polytope P can be preassigned. In Section 4.2 we saw this was always the case if P had no more than $d+3$ vertices. This holds also when $d=3$.

Theorem 6.11 (Barnette-Grünbaum [1970]) *The shape of any facet of any 3-polytope can be preassigned.*

A polytope is *equifacetted* if all its facets are of the same combinatorial type. A d-polytope is *facet-forming* (or a *d-facet*) if it is the combinatorial type of the facets of some equifacetted $(d+1)$-polytope; otherwise it is a *nonfacet*. It is easy to classify 2-polytopes using the condition on p-vectors (see Section 6.7). The triangle, quadrilateral and pentagon are facet-forming, while the n-gon is a nonfacet for every $n \geq 6$. For higher dimensions no classification is known. See Perles and Shephard [1967], Barnette [1980], and Schulte [1985]. Any d-polytope with $d+2$ vertices is facet-forming. Facet-forming polytopes with large numbers of vertices are also known.

Among the equifacetted polytopes are the (combinatorially) regular polytopes, for which vertex-figures are also all of the same combinatorial type. The icosahedron is not yet classified as facet-forming or a nonfacet; all other regular 3-polytopes are known to be facet-forming. The simplex and cube are the only facet-forming regular 4-polytopes (Kalai [1990b]). For general d, the d-simplex and d-cube are, of course, facet-forming; the d-crosspolytope is a nonfacet for $d \geq 4$.

6.6 Barycentric Subdivisions

Let P be a convex d-polytope. Perform a stellar subdivision of P with respect to each of its proper faces in succession, going from high to low dimensional faces. The result is the *barycentric subdivision* of P, $\Delta(P)$, a simplicial d-polytope with vertices corresponding to proper faces of P and faces corresponding to chains of faces of P. As a simplicial complex this is also known as the order complex of the face lattice of P (with least and greatest elements omitted). If each vertex of $\Delta(P)$ is labeled with the dimension of the corresponding face of P, then each facet of $\Delta(P)$ has exactly one vertex with each of the labels $0, 1, \ldots, d-1$.

A simplicial $(d-1)$-complex Δ is *balanced* if, under some labeling of vertices, each facet has one vertex of each label. By connectedness, the labeling is essentially unique in the case that Δ is a balanced sphere. Not all balanced simplicial d-polytopes arise as barycentric subdivisions of polytopes. Those that are barycentric subdivisions of regular CW spheres have been characterized using flag vectors (Bayer [1988]). The definition of flag vectors is extended to balanced simplicial complexes as follows. Let Δ be a balanced simplicial $(d-1)$-complex with vertices labeled by $0, 1, \ldots, d-1$. For each subset $S \subseteq \{0, 1, \ldots, d-1\}$ let $f_S(\Delta)$ be the number of simplices in Δ whose vertices have exactly the labels of S. Note that for a d-polytope P, and the labeling described above for the vertices of $\Delta(P)$, $f_S(\Delta(P))$ agrees with the flag number $f_S(P)$. Thus the numbers $f_S(\Delta(P))$ satisfy the generalized Dehn-Sommerville equations (see Section 3.7). For an arbitrary balanced simplicial polytope Δ the numbers $f_S(\Delta)$ do not necessarily satisfy these equations.

For the characterization of barycentric subdivisions we must go beyond polytopes. Regular CW spheres share some of the combinatorial properties of polytopes. For the definition and motivation for regular CW spheres, see Björner [1984].

Theorem 6.12 (Bayer [1988]) *For any simplicial polytope Δ, Δ is the barycentric subdivision of a regular CW sphere if and only if, for some vertex labeling of Δ, the numbers $f_S(\Delta)$ satisfy the generalized Dehn-Sommerville equations.*

It is an open problem to distinguish barycentric subdivisions of polytopes (or even of polyhedral spheres) among those of regular CW spheres. We know of no example of a p.l. regular CW sphere whose barycentric subdivision is not polytopal.

6.7 p-vectors of 3-polytopes

We conclude our survey with a simple question of the combinatorics of 3-polytopes that remains open. What are the possible distributions of n-gons as facets of 3-polytopes? A partial answer was given a hundred years ago by Eberhard (see Grünbaum [1967]). For a 3-polytope P and an integer $n \geq 3$, let $p_n(P)$ be the number of P's facets that are n-gons. The sequence $(p_n)_{n\geq 3}$ is the *p-vector* of P. Call a sequence $p = (p_3, p_4, p_5, p_7, p_8, \ldots)$ a *reduced (simple) p-vector* if some value of p_6 can be inserted to get the p-vector of some (simple) 3-polytope.

Theorem 6.13 (Eberhard [1891]) *A sequence $(p_3, p_4, p_5, p_7, p_8, \ldots)$ of natural numbers, only finitely many nonzero, is a reduced (simple) p-vector if and only if $\sum_{n\geq 3}(6-n)p_n$ is even and is at least (equal to) 12.*

The values of p_6 that complete a given reduced simple p-vector are now fairly well understood. The following result is from Jendrol' [1983]; it incorporates contributions by various people. For $p = (p_3, p_4, p_5, p_7, p_8, \ldots)$ a reduced simple p-vector, write $\sigma = \sum_{j\neq 6} p_j$, $\rho = \sum_{j\not\equiv 0 \pmod 3} p_j$, and $\mathcal{P}(p) = \{p_6 : p_6 \text{ completes } p \text{ to a simple } p\text{-vector}\}$.

Theorem 6.14 *Let p be a reduced simple p-vector.*

i. If $\rho \leq 2$, then for some integer m, $\mathcal{P}(p)$ contains every integer $k \geq m$ of the same parity as σ and no integer of the opposite parity.

ii. If $\rho \geq 3$, then for some integer m, $\mathcal{P}(p)$ contains every integer $k \geq m$.

In both cases m can be chosen to be at most $\sum_{j\neq 6} jp_j$.

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