

# GRAPHS, SKELETA AND RECONSTRUCTION OF POLYTOPES

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Dedicated to Tibor Bisztriczky, Gábor Fejes Tóth and Endre Makai, on the occasion of their birthdays.

## Abstract

A renowned theorem of Blind and Mani, with a constructive proof by Kalai and an efficiency proof by Friedman, shows that the whole face lattice of a simple polytope can be determined from its graph. This is part of a broader story of reconstructing face lattices from partial information, first considered comprehensively in Grünbaum's 1967 book. This survey paper includes varied results and open questions by many researchers on simplicial polytopes, nearly simple polytopes, cubical polytopes, zonotopes, crosspolytopes, and Eulerian posets.

## 1 Background

This is a survey paper on reconstruction of polytopes, with an emphasis on determining the face lattice of a polytope from its graph or from higher dimensional skeleta. We assume basic familiarity with the combinatorial theory of convex polytopes. For definitions the reader can consult Grünbaum [19] or Ziegler [45]. The reader is also directed to Kalai [29] for a survey of several topics on graphs and polytopes, including reconstruction.

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How much combinatorial information is needed to determine the entire face lattice of a convex polytope? This is the subject of Chapter 12 of Grünbaum's book [19]. In some sense, little information is needed: the vertex-facet incidences of the polytope determine the face lattice. (In fact, for  $d \geq 5$ , the face lattice of a  $d$ -polytope can be reconstructed from the incidences of edges and  $(d - 2)$ -faces [19, Exercise #11 on page 234].) In another sense, a lot of information is needed: the bottom half of the face lattice of a cyclic polytope is the same as the bottom part of a simplex of higher dimension.

In this paper all polytopes are real and convex. We do not actually use the embedding of the polytope in Euclidean space, but we restrict to those that can be embedded. We often blur the distinction between the polytope and its face lattice. Write  $P \cong Q$  to mean that  $P$  and  $Q$  are combinatorially equivalent polytopes, that is, their face lattices are isomorphic.

For  $P$  a  $d$ -polytope and  $k \leq d - 1$ , the  $k$ -skeleton of  $P$  is the subcomplex of the boundary complex of  $P$  consisting of all faces of  $P$  of dimension at most  $k$ . Two polytopes are  $k$ -equivalent if their  $k$ -skeletons are combinatorially equivalent. The polytopes need not be of the same dimension.

**Definition 1** A  $d$ -polytope  $P$  is  $k$ -neighborly if every  $k$ -element subset of the vertices of  $P$  is the vertex set of a face of  $P$ . A  $d$ -polytope is neighborly if it is  $\lfloor d/2 \rfloor$ -neighborly.

Thus a  $k$ -neighborly polytope with  $n + 1$  vertices is  $(k - 1)$ -equivalent to the  $n$ -simplex.

Cyclic polytopes are neighborly polytopes. Thus cyclic  $d$ -polytopes are  $(\lfloor d/2 \rfloor - 1)$ -equivalent to higher dimensional polytopes. In general they can also be  $(\lfloor d/2 \rfloor - 1)$ -equivalent to different  $d$ -polytopes. Grünbaum [19, 7.2.4] constructs an example of a neighborly 4-polytope with 8 vertices that is not combinatorially equivalent to the cyclic 4-polytope with 8 vertices. The construction can be used to produce similar examples with more vertices and in higher dimensions. Padrol [34] gives constructions of many neighborly polytopes.

The following results (due to Gale, Grünbaum, and Perles) can be found in [19] (in the original text or in the additional notes).

**Theorem 1**

1. For  $d \geq 2k + 2$  the  $d$ -simplex is  $k$ -equivalent to the cyclic polytopes of dimensions  $2k + 2, 2k + 3, \dots, d - 1$  with  $d + 1$  vertices.

2. If  $P$  and  $Q$  are  $k$ -equivalent polytopes,  $\dim(P) = d$ , and  $k \geq \lfloor d/2 \rfloor$ , then  $\dim(Q) = d$ .

**Theorem 2** *Assume  $P$  and  $Q$  are  $d$ -polytopes.*

1. If  $P$  and  $Q$  are  $(d - 2)$ -equivalent, then  $P \cong Q$ .
2. If  $P$  and  $Q$  are simplicial and  $P$  and  $Q$  are  $\lfloor d/2 \rfloor$ -equivalent, then  $P \cong Q$ .
3. If  $P$  is a simplex and  $P$  and  $Q$  are 0-equivalent, then  $P \cong Q$ .

The last statement is, of course, a pedantic way of saying that  $d$ -simplices are the only convex  $d$ -polytopes with exactly  $d + 1$  vertices. Part 1 says that any  $d$ -polytope (or, more precisely, its face lattice) can be reconstructed from its  $(d - 2)$ -skeleton. In particular, for  $d = 3$ , the graph of a 3-polytope determines the face lattice; this is a consequence of much earlier results by Steinitz [41] and Whitney [42]. Dancis [12] proves a result analogous to Part 2 for triangulated manifolds. For simplicial  $d$ -polytopes, the face lattice can be reconstructed from the incidences of  $i$ -faces and  $(i + 1)$ -faces, if  $\lfloor d/2 \rfloor < i \leq d - 2$  ([29, Theorem 19.5.25]).

Note in Part 2 of Theorem 2, we need to assume both  $P$  and  $Q$  are simplicial. The following 5-polytopes are 2-equivalent: the bipyramid over the 4-simplex, and the pyramid over the bipyramid over the 3-simplex. The first polytope is simplicial, while the second is “quasisimplicial”—all its facets are themselves simplicial polytopes.

A much stronger (and surprising) result holds for simple polytopes, the duals of simplicial polytopes.

**Theorem 3 (Blind and Mani [11] and Kalai [27])** *If  $P$  and  $Q$  are 1-equivalent simple  $d$ -polytopes, then  $P$  and  $Q$  are combinatorially equivalent.*

That is, the face lattice of a simple polytope is determined by its graph. Note that the theorem assumes  $P$  and  $Q$  are of the same dimension. The graph of a simple  $d$ -polytope may also be the graph of a nonsimple, lower dimensional polytope. For example, the graph of the simplex is also the graph of lower dimensional cyclic polytopes; the graph of the cube is also the graph of lower dimensional “neighborly cubical polytopes”. The dual statement of the theorem is that the face lattice of a simplicial polytope can be reconstructed from its facet-ridge graph.

Kalai’s proof gives a method for constructing the face lattice from the graph, but the complexity is exponential in the number of vertices—the

algorithm uses all acyclic orientations of the graph. It is straightforward, however, to construct the face lattice of a simple polytope from its 2-skeleton. (See [27].) Murty [33] shows that the 2-skeleton is enough to determine the face poset of a broader class of objects, “simple” *abstract polytopes*. (An abstract or *incidence polytope*, introduced by Danzer and Schulte [13], is a strongly flag-connected graded poset with the diamond property. Earlier, Adler [4] had given a similar definition, but with a further condition that each vertex is contained in exactly  $d$  facets.) Joswig [22] shows that the face lattice of any polytope  $P$  can be reconstructed from its graph along with the following additional information: for each vertex  $v$  the sets of edges that are precisely the edges containing  $v$  and contained in a facet of  $P$ .

Various papers consider algorithmic issues of constructing the face lattice of a simple polytope from its graph ([1, 23, 26]; Friedman [18] gives a polynomial time algorithm for finding the facets of the simple polytope from the graph.

Results of Perles show the limitations of polytope reconstruction theorems, even for polytopes with few vertices: there are many more combinatorial types of  $d$ -polytopes with  $d + 3$  vertices than combinatorial types of  $k$ -skeleta of these polytopes.

**Theorem 4 (Perles, see [19])** *The number of combinatorial types of  $d$ -polytopes with  $d + 3$  vertices is bounded below by an exponential function of  $d$ .*

**Theorem 5 (Perles, see Part II of [43], also [28])** *For fixed  $k$  and  $b$ , the number of combinatorial types of  $k$ -skeleta of  $d$ -polytopes with  $d + b + 1$  vertices is bounded above by a constant independent of dimension  $d$ .*

## 2 Nearly simple polytopes

Recent papers by Doolittle, et al. [15] and by Pineda-Villavicencio, et al. [36] consider the possibility of extending the result on simple polytopes to polytopes that have few nonsimple vertices (vertices of degree greater than the dimension of the polytope).

**Theorem 6 ([15, 36])**

1. *The face lattice of any  $d$ -polytope with at most two nonsimple vertices is determined by its graph.*

2. If  $d \geq 5$  and  $n \leq 2d$ , then the face lattice of any  $d$ -polytope with  $n$  vertices, at most  $2d - n + 3$  of them nonsimple, is determined by its graph.
3. The face lattice of any  $d$ -polytope with at most  $d - 2$  nonsimple vertices is determined by its 2-skeleton.
4. For every  $d \geq 4$ , there are two combinatorially nonequivalent  $d$ -polytopes with  $d - 1$  nonsimple vertices and isomorphic  $(d - 3)$ -skeleton.

In terms of  $k$ -equivalence, this says:

If a  $d$ -polytope  $P$  has at most two nonsimple vertices, and a  $d$ -polytope  $Q$  is 1-equivalent to  $P$ , then  $P \cong Q$ .

If a  $d$ -polytope  $P$  has at most  $d - 2$  nonsimple vertices, and a  $d$ -polytope  $Q$  is 2-equivalent to  $P$ , then  $P \cong Q$ .

There are  $(d - 3)$ -equivalent  $d$ -polytopes with  $d - 1$  nonsimple vertices that are not combinatorially equivalent.

The paper [15] presents two proofs of Part 1 of the theorem. One uses the acyclic orientation approach of Kalai's proof [27] of the simple case and the "frames" of Joswig, Kaibel and Körner [23]. (See also [18, 26].) The other uses acyclic orientations along with truncation of the nonsimple vertices.

The construction in [15] of Part 4 of Theorem 6 gives a 4-polytope with  $f$ -vector  $(8, 19, 18, 7)$ , one of whose facets is the bipyramid over a triangle. Splitting this facet into two simplices gives a complex combinatorially equivalent to a 4-polytope with  $f$ -vector  $(8, 19, 19, 8)$ . The construction does not change the graph and therefore does not change the number (three) of nonsimple vertices.

Pineda-Villavicencio et al. [36, 37] also consider another measure of deviation from simple. The *excess degree* of a  $d$ -polytope is  $2f_1 - df_0$ , the sum of the number of extra edges on all the vertices. They study Minkowski decomposability and prove some structural properties of polytopes with small excess. On the issue of reconstruction, they prove the following theorem.

**Theorem 7 ([36])** *The face lattice of any  $d$ -polytope with excess degree at most  $d - 1$  is determined by its graph.*

### 3 Crosspolytopes and Centrally Symmetric Polytopes

Espenschied [16] asked for the dimensions of polytopes that are 1-equivalent to the  $d$ -crosspolytope. The  $d$ -dimensional crosspolytope can be obtained by starting with the 1-polytope and successively taking bipyramids  $d - 1$  times. The graph of the  $d$ -crosspolytope is the complete  $d$ -partite graph with 2 vertices in each part, denoted  $K_{2,2,\dots,2}$ . This graph does not uniquely determine the polytope, in general. For example, one can take the convex hull of the 3-crosspolytope in  $\mathbf{R}^4$  along with a segment intersecting the 3-crosspolytope only at the interior point of one of its facets. The result is a 4-polytope 1-equivalent to the 4-crosspolytope, having one facet a triangular bipyramid. In what dimensions other than  $d$  do there exist polytopes 1-equivalent to the  $d$ -crosspolytope?

**Theorem 8 (Espenschied [16])** *Let  $d$  be an integer,  $d \geq 4$ .*

1. *If  $P$  is a polytope that is 1-equivalent to the  $d$ -crosspolytope, then  $\dim(P) \leq \lfloor 3d/2 \rfloor - 1$ .*
2. *For every  $n$ ,  $d \leq n \leq \lfloor 3d/2 \rfloor - 1$ , there exists an  $n$ -dimensional polytope that is 1-equivalent to the  $d$ -crosspolytope.*
3. *If  $d \geq 5$ , there exists a  $(d-1)$ -dimensional polytope that is 1-equivalent to the  $d$ -crosspolytope.*

(Part 3 is derived from an example of [19]; see below.)

Part 2 generalizes to  $k$ -equivalence.

**Theorem 9** *Let  $k$  be a positive integer,  $d \geq 2(k+1)$ . For every  $n$ ,  $d \leq n \leq \lfloor \binom{k+2}{k+1}d \rfloor - 1$ , there exists an  $n$ -dimensional polytope that is  $k$ -equivalent to the  $d$ -crosspolytope.*

The proof of Part 2 of Theorem 8 follows from Espenschied's observation that by taking joins of a sequence of crosspolytopes, one obtains polytopes 1-equivalent to a crosspolytope. The idea extends to give Theorem 9 as well.

**Definition 2** The *join*  $P*Q$  of  $d$ -polytope  $P$  and  $e$ -polytope  $Q$  is the convex hull of copies of  $P$  and  $Q$  embedded in skew affine subspaces of  $\mathbf{R}^{d+e+1}$ .

Denote the  $j$ -dimensional crosspolytope by  $X_j$ . For integers  $j_1, j_2, \dots, j_s$ ,  $j_i \geq k+1$ , let  $d = j_1 + j_2 + \dots + j_s$ . The join  $X_{j_1} * X_{j_2} * \dots * X_{j_s}$  is a polytope

of dimension  $d + s - 1$ , and is  $k$ -equivalent to the crosspolytope  $X_d$ . This construction gives Theorem 9 and, in particular, Part 2 of Theorem 8. Note that the graph of the crosspolytope is  $d$ -colorable, so any  $n$ -polytope with  $n > d$  that is 1-equivalent to  $X_d$  can have no simplex faces of dimension greater than  $d - 1$ .

The regular  $d$ -crosspolytope is centrally symmetric; it is clearly the centrally symmetric  $d$ -polytope with the fewest vertices. Grünbaum [19, Sec. 6.4] gives the following example of a centrally symmetric 4-polytope with 10 vertices and 40 edges. This 4-polytope has vertices  $\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm(e_1 + e_2 + e_3 + e_4)$ , and has the graph of the 5-crosspolytope. By taking successive bipyramids over Grünbaum's 4-polytope, one obtains the polytopes of Theorem 8, Part 3. Espenschied observes that the join of Grünbaum's 4-polytope with a square is a 7-dimensional polytope that is 1-equivalent, but not combinatorially equivalent, to the 7-crosspolytope; it is, after all, nonsimplicial.

A centrally symmetric polytope  $P$  is *centrally symmetric  $k$ -neighborly* if every set of  $k$  vertices of  $P$ , no two of which are antipodes, is the vertex set of a  $(k - 1)$ -face of  $P$ . The  $d$ -crosspolytope is centrally symmetric  $k$ -neighborly for all  $k \leq d$ . Thus a centrally symmetric polytope with  $2d$  vertices is  $k$ -neighborly if and only if it is  $(k - 1)$ -equivalent to the  $d$ -crosspolytope. McMullen and Shephard [32] introduced centrally symmetric diagrams (analogues of Gale diagrams) to study centrally symmetric polytopes with few vertices. They used these to construct  $d$ -polytopes that have  $2(d + 2)$  vertices and are centrally symmetric  $k$ -neighborly, for  $k \approx d/3$ . Barvinok, Lee and Novik [8] construct centrally symmetric  $k$ -neighborly  $n$ -polytopes where  $n$  is small relative to the number of vertices, that is, small relative to the dimension  $d$  of the  $(k - 1)$ -equivalent  $d$ -crosspolytope. In particular, they give polytopes of dimension approximately  $2 \log_3(2d)$ , 1-equivalent to the  $d$ -crosspolytope. See also [10] for construction of centrally symmetric neighborly spheres.

Joswig and Ziegler [25] ask if for every  $n$ ,  $4 \leq n \leq d$  and  $k = \lfloor n/2 \rfloor - 1$  there is an  $n$ -polytope  $k$ -equivalent to the  $d$ -crosspolytope. Linial and Novik [30] prove the existence of  $k$ -neighborly centrally symmetric  $n$ -polytopes with  $2d$  vertices, that is,  $n$ -polytopes  $(k - 1)$ -equivalent to the  $d$ -crosspolytope, with asymptotic estimates for  $k$  in terms of  $n$  and  $d$ . (See also Donoho [14].)

Recall that the graph of the crosspolytope is the complete multipartite graph with all parts of size 2. What other complete multipartite graphs are the graphs of polytopes?

**Proposition 10 (Espenschied [16])** *For  $1 \leq i \leq t$ , let  $n_i \in \{1, 2\}$ . The*

complete multipartite graph  $K_{n_1, n_2, \dots, n_t}$  ( $t > 1$ ) is the graph of a polytope if and only if  $\{n_1, n_2, \dots, n_t\}$  is not one of the multisets  $\{1, 2\}$  or  $\{1, 1, 2\}$ .

These graphs can all be realized with iterated pyramids and bipyramids.

Espenschied guessed that these were the only complete multipartite graphs realizable as graphs of polytopes. It is easy to check, using the nonplanarity of  $K_{3,3}$ , that  $K_{m,n}$  is not the graph of any polytope, when  $m$  and  $n$  are both at least 3 [7]. However, Firsching [17] has found examples showing Espenschied's guess is incorrect. He has examples of 4-polytopes with 9 vertices with the following graphs:  $K_{3,2,2,2}$ ,  $K_{3,2,2,1,1}$ ,  $K_{3,2,1,1,1,1}$  and  $K_{3,1,1,1,1,1,1}$ . Zheng [44] constructs simplicial 3-spheres with graph  $K_{4,4,4,4}$ , but they are believed not to be polytopal.

## 4 Cubical Polytopes and Zonotopes

Perhaps the first thing to note about cubical polytopes is that the results about crosspolytopes can be dualized to results about cubes. A polytope 1-equivalent to a  $d$ -crosspolytope then corresponds to a polytope that shares the facet-ridge graph with a cube.

**Definition 3** A polytope is *cubical* if and only if all of its proper faces are combinatorially equivalent to cubes.

**Theorem 11 (Joswig and Ziegler [25])**

1. For  $n \geq d \geq 2k + 2$ , there exists a cubical  $d$ -polytope  $k$ -equivalent to the  $n$ -dimensional cube.
2. If a  $d$ -polytope  $P$  is  $k$ -equivalent to the  $n$ -cube for  $k \geq d/2$ , then  $P$  is a  $d$ -cube.

Cubical  $d$ -polytopes that are  $(\lfloor d/2 \rfloor - 1)$ -equivalent to an  $n$ -dimensional cube (for some  $n \geq d$ ) are called *neighborly cubical* polytopes. A neighborly cubical  $d$ -polytope may not be reconstructible from its  $(\lfloor d/2 \rfloor - 1)$ -skeleton, however. Joswig and Ziegler [25] give an example of a 4-polytope that is 1-equivalent to a 5-cube (and thus to a neighborly cubical 4-polytope), but that is not itself cubical. (It has a facet that can be subdivided into two 3-cubes.) See also [6, 24].

Joswig [22] shows that a certain class of cubical polytopes, “capped cubical polytopes,” can be reconstructed from their graphs.

Another generalization of the cube is the zonotope, the Minkowski sum of 1-polytopes.

### Theorem 12

1. (Björner, Edelman and Ziegler [9]) *Zonotopes are determined by their graphs.*
2. (Babson, Finschi and Fukuda [5]) *Duals of cubical zonotopes are determined by their graphs.*

## 5 Eulerian Posets

In the previous sections we have seen conditions under which skeletal information about a polytope enables us to reconstruct the entire face lattice. What if we consider face lattices of polytopes within the larger class of Eulerian posets?

**Definition 4** An *Eulerian poset* is a graded, finite partially ordered set such that in each interval, the number of elements of even rank equals the number of elements of odd rank.

The face lattice of a  $d$ -polytope is an Eulerian poset of rank  $d + 1$ . In the mixed company of polytopes and posets, there is always the difficulty of choosing between dimensions and ranks. Here the dimension perspective will prevail. For a rank  $d + 1$  Eulerian poset  $Q$ , write  $Q_k$  for the set of rank  $k + 1$  elements (also called dimension  $k$  elements).

Suppose we change the hypothesis of Theorem 2 from “Assume  $P$  and  $Q$  are  $d$ -polytopes” to “Assume  $P$  is a  $d$ -polytope and  $Q$  is a rank  $d + 1$  Eulerian poset.” Do we get a theorem? No, not even for Part 3 of the theorem. Figure 1 shows an Eulerian poset of rank 4 with four atoms that is not the Boolean algebra (face lattice of a 3-simplex). It also serves to illustrate the proof of Theorem 13.

This generalizes to a construction of an Eulerian poset of rank  $d + 1$  that is  $(d - 3)$ -equivalent to the  $d$ -simplex. In fact, it generalizes much further.

**Theorem 13** *If  $P$  is a  $d$ -polytope,  $d \geq 3$ , and  $j$  is an integer,  $0 \leq j \leq d - 1$ , then there exists a rank  $d + 1$  Eulerian poset, not isomorphic to the face lattice of  $P$ , but which differs from the face lattice of  $P$  only in dimensions  $j$  and  $j + 1$ .*

**Proof:** Choose a  $j$ -face  $F$  of  $P$  and a  $(j + 1)$ -face  $G$  containing  $F$ . Construct a poset  $Q$  whose elements are all the elements of the face lattice of  $P$  along with four new elements  $A_1, A_2, B_1$  and  $B_2$ . The pairs in the poset are those pairs in the face lattice of  $P$  along with the following pairs:

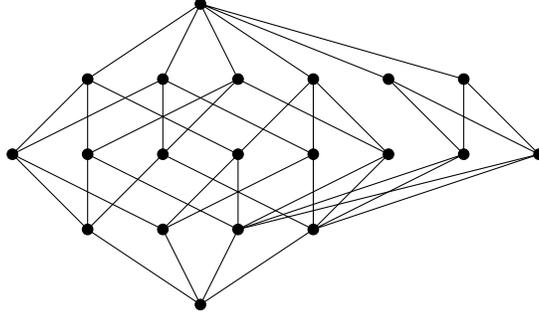


Figure 1: An Eulerian poset 0-equivalent to the 3-simplex

- $(A_i, B_\ell)$  for  $i, \ell \in \{1, 2\}$
- $(C, A_i)$  and  $(C, B_\ell)$ , for  $i, \ell \in \{1, 2\}$  and  $C$  a proper face of  $F$
- $(A_i, D)$  and  $(B_\ell, D)$  for  $i, \ell \in \{1, 2\}$  and  $D$  a face of  $P$  properly containing  $G$ .

(See Figure 1.) By construction, the elements  $A_i$  are of dimension  $j$ , the elements  $B_\ell$  are of dimension  $j + 1$ , and  $Q$  agrees with the face lattice of  $P$  except at dimensions  $j$  and  $j + 1$ . It is straightforward to check the interval condition to show that  $Q$  is Eulerian. We illustrate with one of the more interesting cases, an interval of the form  $[C, B_1]$ , where  $C$  is a proper face of  $F$ . Since the face lattice of  $P$  is Eulerian, the interval  $[C, F]$  in  $P$  has the same number of elements of even and odd rank. The interval  $[C, B_1]$  of  $Q$  has all elements of  $[C, F]$  except  $F$ , has two other elements ( $A_1$  and  $A_2$ ) of the same rank as  $F$ , and has one other element ( $B_1$ ) of rank one more. So the interval  $[C, B_1]$  of  $Q$  has the same number of elements of even and odd rank.  $\square$

Thus we know that Parts 2 and 3 of Theorem 2 fail in the generality of Eulerian posets. Part 1 fails even for  $P$  a simplicial polytope (but not for  $P$  a simplex) and  $Q$  an Eulerian poset. Consider the  $d$ -crosspolytope. The  $2^d$  facets are naturally partitioned into two sets, so that no two facets in the same set intersect at a ridge ( $(d - 2)$ -face). Remove one of the sets of facets, and duplicate the other set. The resulting poset is an Eulerian poset with the same  $(d - 2)$ -skeleton as the crosspolytope.

However, in the case of a simplicial polytope, we can reconstruct a single missing rank of lower rank (dimension less than  $d - 1$ ). Write  $P \setminus P_r$  for

the subposet of  $P$  consisting of all the elements of ranks other than  $r + 1$  (dimension  $r$ ).

**Theorem 14** *Let  $d \geq 3$ , and assume  $P$  is a  $d$ -polytope and  $Q$  is a rank  $d + 1$  Eulerian poset.*

1. *If  $P$  is simplicial,  $0 \leq r \leq d - 2$ , and  $P \setminus P_r \cong Q \setminus Q_r$ , then  $P \cong Q$ .*
2. *If  $P$  is simple,  $1 \leq r \leq d - 1$ , and  $P \setminus P_r \cong Q \setminus Q_r$ , then  $P \cong Q$ .*
3. *If  $P$  is a simplex,  $0 \leq r \leq d - 1$ , and  $P \setminus P_r \cong Q \setminus Q_r$ , then  $P \cong Q$ .*

## 6 Related Issues on Graphs and $k$ -Skeleta

We review briefly some other work on graphs and  $k$ -skeleta of polytopes.

A relatively easy way of reconstructing simple polytopes from their graphs would have followed from the truth of a conjecture by Perles: The facet subgraphs of a simple  $d$ -polytope are exactly all the  $(d - 1)$ -regular, connected, induced, nonseparating subgraphs of the graph of the polytope. However, the conjecture is false (even when the subgraph is required to be  $(d - 1)$ -connected), as shown by Haase and Ziegler [20]. In the known 4-dimensional counterexamples the offending subgraphs are not planar, so they ask if adding planarity is enough to guarantee a facet of a 4-polytope. This extends to higher dimensions by asking for the following. Are the facet subgraphs of a simple  $d$ -polytope exactly all the  $(d - 1)$ -regular, induced, nonseparating subgraphs that are isomorphic to the graph of a  $(d - 1)$ -polytope? However, since there is no easy characterization of graphs of higher dimensional polytopes, this would not be an effective characterization. Adiprasito, Kalai and Perles [2] suggested that “isomorphic to the graph of a  $(d - 1)$ -polytope” could be weakened to “isomorphic to the graph of a homology  $(d - 2)$ -sphere.”

We have seen definitions of neighborly, centrally symmetric neighborly, and neighborly cubical polytopes. All can be defined in terms of  $k$ -equivalence with familiar polytopes of higher dimensions. Here we mention two other such classes of polytopes. A *prodsimplicial-neighborly* polytope [31] is a polytope with the same  $k$ -skeleton (for specified  $k$ ) as the Cartesian product of simplices. Matschke et al. [31] construct prodsimplicial-neighborly polytopes, including polytopes of dimension  $2k + r + 1$  that are  $k$ -equivalent to the product of  $r$  simplices. The latter are Minkowski sums of cyclic polytopes, and are examples of *Minkowski neighborly* polytopes, studied in [3].

Several examples mentioned above (including cubical neighborly polytopes) are created by projection from higher dimensional polytopes. As another example, Sanyal and Ziegler [39] construct  $d$ -dimensional projections of polytopes that are  $(\lfloor d/2 \rfloor - 1)$ -equivalent to the  $r$ -fold product of  $m$ -gons, for every even  $m \geq 4$  and every  $d \leq 2r$ . However, in general it is not clear when the projection of a polytope preserves its  $k$ -skeleton. Rörig and Sanyal [38] study obstructions to the existence of projections of polytopes that preserve  $k$ -skeleta.

So far we have always assumed that the graph or  $k$ -complex under consideration is known to be the graph or  $k$ -skeleton of a polytope. However, we have no characterization of graphs of polytopes of dimensions four and higher. Pfeifle, Pilaud and Santos [35] review necessary conditions for a graph to be polytopal, construct families of graphs that satisfy these conditions but are not polytopal, and investigate polytopality of products of graphs. In particular, they show the following theorem.

**Theorem 15 ([35])** *The Cartesian product of graphs is the graph of a simple polytope if and only if each of its factors is the graph of a simple polytope.*

## 7 Open Problems

We are left with many open problems. The first is a conjecture of Grünbaum.

**Conjecture 16 (Grünbaum [19])** *If a  $k$ -dimensional complex  $\mathcal{C}$  is the  $k$ -skeleton of both a  $d$ -polytope and a  $d'$ -polytope, where  $d \leq d'$ , then for every  $d', d \leq d' \leq d''$ , there is a  $d'$ -polytope having  $\mathcal{C}$  as its  $k$ -skeleton.*

Of course, we can dream and ask: for each  $k$ , characterize the polytopes whose  $k$ -skeleta determine the face lattice. Here are some less ambitious questions.

- ([15]) For  $d \geq 5$  does there exist an integer  $j$  such that every  $d$ -polytope with  $j$  nonsimple vertices is determined by its 2-skeleton, but not every  $d$ -polytope with  $j$  nonsimple vertices is determined by its 1-skeleton?
- For  $k > 1$  does there exist an  $n$ -dimensional polytope that is  $k$ -equivalent to the  $d$ -crosspolytope for  $\lfloor (\frac{k+2}{k+1})d \rfloor \leq n \leq \lfloor 3n/2 \rfloor - 1$ ? The proof of the upper bound  $\lfloor 3n/2 \rfloor - 1$  for  $k = 1$  (Part (1) of Theorem 8) is based on the fact (proved by Halin [21]) that the Hadwiger number of the graph  $K_{2,2,\dots,2}$  is  $\lfloor 3n/2 \rfloor$ . Is there an analogous theory for  $k > 1$ ?

- Does there exist a  $d$ -polytope  $P$  (not simple or simplicial), an Eulerian poset  $Q$ , and an  $r$ ,  $1 \leq r \leq d - 2$ , such that  $Q$  agrees with the face lattice of  $P$  everywhere except at dimension  $r$ ? Note that the number of elements of  $P_r$  is determined by Euler's formula.
- It seems that for any reasonable reconstruction results involving polytopes and Eulerian posets we will need to restrict to Eulerian lattices. What are the best reconstruction results in this case?
- What reconstruction results can we get for the abstract polytopes of Danzer and Schulte [13]? See [40, Section 7] for a discussion of problems about skeleta of abstract polytopes.

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